Group Work 1, Section 5.2
Cookie Dough Joe

We say that $P : \mathbb{R}^n \to W$ is a ***projection*** if $P \circ P = P^2 = P$; that is, if $P\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in W$. Thus a projection onto W is a map that "fixes" W. Of course, an orthogonal projection is a special type of projection. We will now outline the recipe for constructing projections and then you will have a chance to build one of you own. Here are the steps:

I Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for W in $\mathbb{R}^n$. (Note we do not require any orthogonality.)

II Here is the place that requires work: choose $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ so that $\mathbf{v}_i \cdot \mathbf{u}_j = \delta_{ij}$ (here we use $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$). This will involve solving an (underdetermined) linear system, so there will be a non-unique selection of such $\mathbf{u}_j$'s.

III Here is the way we define a projection onto W relative to basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$:

$$P\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\,\mathbf{v}_1 + \cdots + (\mathbf{x} \cdot \mathbf{u}_k)\,\mathbf{v}_k$$

Now for some examples.

1. Show that the construction above does in fact yield a projection.
Hint: Show that $P\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in W$.

2. In $\mathbb{R}^3$, let W be the subspace with basis $\mathbf{v}_1 = [1, 1, 0]$ and $\mathbf{v}_2 = [1, 0, 0]$. Let's find all possible $\mathbf{u}_1$, $\mathbf{u}_2$ such that $\mathbf{v}_i \cdot \mathbf{u}_j = \delta_{ij}$. Initially we have 6 unknowns (3 coordinates for $\mathbf{u}_1$ and 3 for $\mathbf{u}_2$); the δ_{ij} requirements yield 4 equations, and so we expect 2 free parameters associated with $\mathbf{u}_1$ and $\mathbf{u}_2$. Now find this family of $\mathbf{u}_1$, $\mathbf{u}_2$ pairs.

3. Let $\mathbf{u}_1 = [0, 1, s]$ and $\mathbf{u}_2 = [1, -1, t]$. How should s and t be chosen so that $P\mathbf{e}_3 = P\,[0, 0, 1] = [0, 1, 0]$?

4. Find s and t such that $\mathbf{x} - P\mathbf{x}$ is orthogonal to W.