Homework 6: More on Lines and Planes
Math 224
Professor Smith

This worksheet is intended to help you deepen your understanding of the meaning of vector and normal forms of equations of lines and planes. (Note that general and parametric forms consist of writing out the specific equations that result from the vector and normal forms, so if you understand the latter, you also understand the former).

1. Let $\mathbf{d} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The line ℓ passing through the point $(2, 0)$ and parallel to the vector $\mathbf{d}$ has vector form
$$\mathbf{x} = t\mathbf{d} + \mathbf{p}.$$
   (a) Choose two positive (small) values of t and two negative values of t.

   (b) For each value of t that you chose in part (a), draw the corresponding vector $\mathbf{x}$ whose terminal point (when in standard position) lies on ℓ. Label each $\mathbf{x}$ with the appropriate value of t and draw each $\mathbf{x}$ as a vector sum.

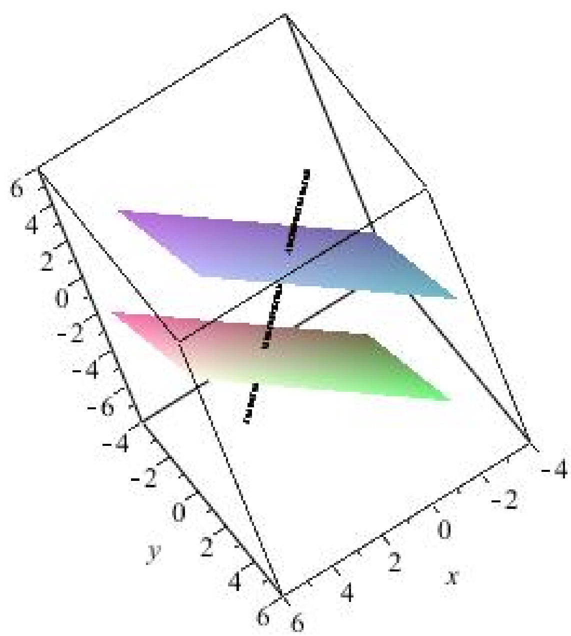
2. Let $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The line ℓ (from above) can also be described by the equation
$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p},$$
or equivalently,
$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0.$$
Consider the following points on ℓ:
$$p_1 = (0, 4), p_2 = (1, 2), p_3 = (3, -2), p_4 = (5, -6).$$
For each point, associate to it a vector which, when in standard position, has terminal point equal to p_i :
$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}.$$
   (a) Draw each $\mathbf{x}_i$ on a graph with the line ℓ. Then draw the vector $\mathbf{x}_i - \mathbf{p}$ to see that in each case, the resulting vector is orthogonal to $\mathbf{n}$.

   (b) Show algebraically that each vector $\mathbf{x}_i - \mathbf{p}$ is orthogonal to $\mathbf{n}$.

Before you begin Problem 3, let's make a couple observations.
(i) The idea we discussed in class (namely, that we'd like the length of something like $\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}$, is an idea that we will return to throughout the course because it is useful and it generalizes nicely. Unfortunately, for the problem you want to complete, you

don't have one of the necessary tools: specifically, you don't know how to project a vector onto a plane. The good news is that, eventually, you will. For now, though, we have to approach this in a different way (see Observation (ii)).

(ii) In order to succeed with this problem, we need to draw ourselves a good picture. From the figure below, we notice that we don't want the length of just any vector between the planes; rather, we need the length of a portion of normal vector.



With some thought, it is hopefully clear that what you need to do is the following. Get a vector $\mathbf{v}$ that takes a point P on plane one to a point Q on plane two. Project $\mathbf{v}$ onto the normal vector (I'm using the phrase "the normal vector" because any normal vector for one plane is also a normal vector for the other plane, so choose one and make it your normal vector). Now find the length of your projection. Make sense?

3. Find the distance between the parallel planes described by the equations below.

$$2x + y - 2z = 0 \text{ and } 2x + y - 2z = 5.$$