Local Extrema

Math 213, Calculus III

Parts of this worksheet are adapted from a worksheet created by Prof. Carol Schumacer.

Name:______________________________

In Sections 15.1 and 15.2, we explore local and global extrema of functions of several variables. Let's begin with the definitions, which are natural generalizations of the definitions you know for functions of one variable.

- A function f has a **local maximum** at the point P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.
- A function f has a **local minimum** at the point P_0 if $f(P_0) \leq f(P)$ for all points P near P_0.
- A function f has a **global maximum on the region** R at the point P_0 if $f(P_0) \geq f(P)$ for all points P in R.
- A function f has a **global minimum on the region** R at the point P_0 if $f(P_0) \leq f(P)$ for all points P in R.

1. Use the graphs on page 617 for question 14 to find an example of each of the following. List the graph number as well as either a rough estimate of the coordinates of the point or a description of the location of the point. If no such point exists, then briefly explain why not.

   (a) A point that is a local maximum and a global maximum.

   (b) A point that is a local minimum and a global minimum.

   (c) A point that is a local maximum but not a global maximum.

   (d) A point that is a local minimum but not a global minimum.

Finding Local Extrema

2. We now turn our attention to finding and classifying local extrema. Suppose that at a point Q, the gradient vector for a function f is defined and nonzero.

   (a) If you were to move away from Q in the direction of $\text{grad} f(Q)$, what would happen to the function f?

   (b) If f has a local maximum at a point P_0 and $\text{grad} f(P_0)$ is defined, what must be true about $\text{grad} f(P_0)$?

   (c) If you were to move away from Q in the direction *opposite* $\text{grad} f(Q)$, what would happen to the function f?

   (d) If f has a local minimum at a point P_0 and $\text{grad} f(P_0)$ is defined, what must be true about $\text{grad} f(P_0)$?

   (e) Give an example (either algebraically, graphically, or by description) of a function that has a local max or min at a point P_0 at which $\text{grad} f(P_0)$ is undefined.

Given these observations, it is natural to define critical points in the following way.

- Points where the gradient is either $\vec{0}$ or undefined are called **critical points** of the function.

The *only* points at which a function f can have a local maximum or minimum are critical points and points on the boundary of its domain. Hence, if we want to find local extrema, we start by finding all critical points (and boundary points if applicable).

Classifying Local Extrema

3. Once we have found all critical points of a function, we must determine whether these points are local maxima, local minima, or neither. A nice technique for doing so is given by the following test.

   The Second Derivative Test for Functions of Two Variables
   Suppose (x_0, y_0) is a point where $\text{grad} f(x_0, y_0) = \vec{0}$. Let

   $$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

   - If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
   - If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
   - If $D < 0$, then f has a saddle point at (x_0, y_0).
   - If $D = 0$ anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of these, at (x_0, y_0).

   The function D is called the *discriminant* of f. We are interested in whether $D(x_0, y_0)$ is positive, negative, or zero. Before we apply the second derivative test to some examples, let's understand why it works.

4. Consider first the case when $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) < 0$.

   (a) What can you say about the sign of $D(x_0, y_0)$? What does the second derivative test tell you in this case?

   (b) Explain graphically why this result makes sense.

5. Consider now the case when $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) > 0$.

   (a) What does $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) > 0$ tell you about the graph if $f_{xx}(x_0, y_0) > 0$?

   (b) What does $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) > 0$ tell you about the graph if $f_{xx}(x_0, y_0) < 0$?

Now comes the really big question: What role does $(f_{xy}(x_0, y_0))^2$ play?

6. Consider the function $f(x, y) = 4xy + x^2 + y^2$.

   (a) Verify that the origin is a critical point for f.

   (b) Now verify that $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ are both positive. What does this allow you to conclude about the x and y cross-sections at the origin?

   (c) Open, execute, and read through the Maple file SecondDerivativeTest.mw. What is the nature of the critical point at the origin?

7. Now that we have a good understanding of the second derivative test, let's try some examples! For each of the following functions,

   - Find the critical points of f.
   - Find the discriminant of f.
   - Use the second derivative test to classify each of the critical points you found.

   Use the graphs provided in the Maple file LocalExtrema.mw to visualize the functions and check your work.

   (a) $f(x, y) = 2x^3 - 24xy + 16y^3$

(b) $f(x, y) = 2x^2 + y^2 + \frac{2}{x^2 y}$

(c) $f(x, y) = 4x^2 + 12xy + 9y^2 + 25$