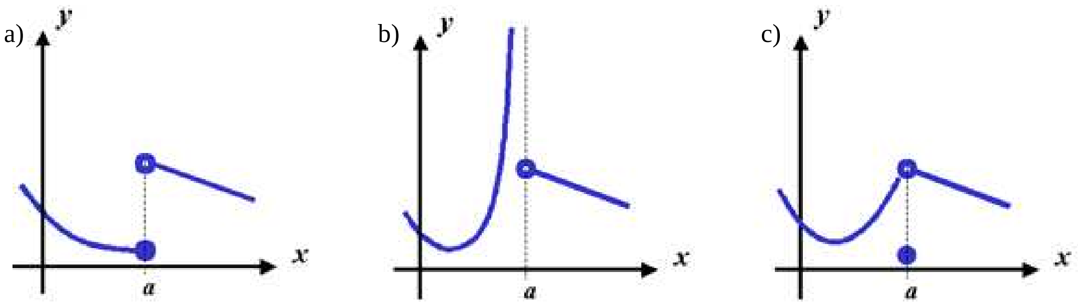
Limits of Functions of Two Variables

Before we discuss limits of functions of two variables, let us review the notion of limits in the single variable context. Recall that the limit of a function $f(x)$ at a point a exists if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$ for some real number L. Hence, to determine whether or not $\lim_{x \to a} f(x)$ exists, we simply compute the limit from the left and the limit from the right, and then check to see if the two limits are equal. If either of the limits fails to exist or if they exist but are not equal, then the limit $\lim_{x \to a} f(x)$ does not exist.

Figure 1. a) The limit does not exist at $x = a$ because the limit from the left does not equal the limit from the



right. b) The limit does not exist at $x = a$ because the limit from the left does not exist. c) The limit at $x = a$ does exist in this case. The limits from the left and right exist, and they are equal.

Limits of functions of two variables are defined very much like functions of one variable. However, the additional dimension can sometimes lead to unexpected results. As you will soon see, the interesting behavior arises because there are infinitely many different ways to approach a point in the plane. In order for the limit $\lim_{(x,y) \to (a,b)} f(x, y)$ to exist, the limits *along all possible paths* into (a, b) must exist and be equal

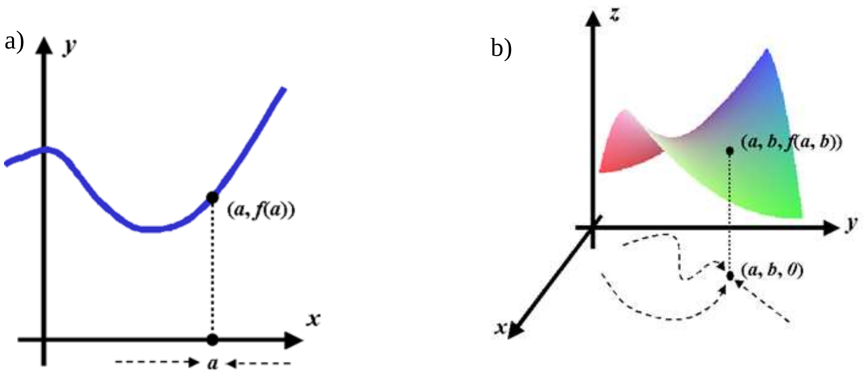


Figure 2. a) There are only two paths of approach when taking the limit of a function of one variable. b) There are infinitely many ways to approach a point when considering limits of functions of two (or more) variables.

Adapted from a lab created by Prof. Judy Holdener.

Example 1 (A Pathological Example).

As our first example, we consider the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, $(x, y) \neq (0,0)$. This is a good example to consider because of the quirky behavior exhibited near the origin. Open the Maple file: **Limits.mw**, and take a look at the plot of the surface $z = f(x, y)$ together with its contour diagram. Then examine the limits along selected pathways into (0, 0) as demonstrated in the file. Finally, compute (by hand) the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ along the line $y = mx$, treating m as a constant. What does your computation indicate? What do you conclude about the limit $\lim_{(x,y)\to(0,0)} f(x,y)$?

The strange behavior exhibited by the function in the previous example is not typical of the sorts of functions that tend to arise in practice. Nonetheless, we examine "pathological" examples like the one above, because they are invaluable in clarifying the concept of a limit. A more typical situation is illustrated by the "straightforward" example below.

Example 2 (A Straightforward Example)

Consider the function $g(x, y)$ obtained by adding one to the denominator of the function $f(x, y)$ defined in the previous example. That is, $g(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2 + 1}$. Note that this seemingly small change to $f(x, y)$ creates a function that is well-defined at the origin. Examine the limit of the $g(x, y)$ at the origin by working through Example 2 of **Limits.mw**. Then compute $\lim_{(x,y)\to(0,0)} g(x,y)$ along the line $y = mx$ by hand. What can you conclude from your computation?

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Example 3 Consider the function $f(x, y) = \dfrac{2x^2 \cdot y}{x^4 + y^2}$ and its limit at the origin. What limit do you get if you approach the origin along straight lines $y = mx$? What if you approach the origin along parabolas? What lesson do you learn from this example?

Example 4. Consider the function $f(x, y) = \dfrac{3x^2 \cdot y}{x^2 + y^2}$ at the origin. Determine the limit along the straight lines to the origin, as well as parabolas. What can you conclude about this limit?

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Defining the Limit Rigorously…

What does it mean for $f(x, y)$ to have a limit at (a, b)?

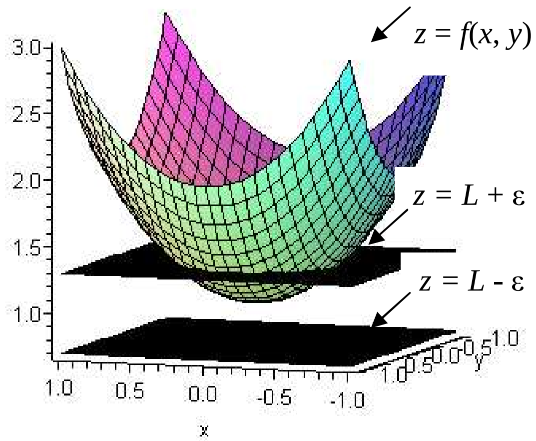
The function f has a limit L at a point (a, b), written $\lim_{(x,y)\to(a,b)} f(x,y)$, if the difference $|f(x,y) - L|$ is as small as we wish whenever the distance from the point (x, y) to the point (a, b) is sufficiently small, but not zero.

How do we say this formally?

The function f has a limit L at a point (a, b), written $\lim_{(x,y)\to(a,b)} f(x,y)$, if whenever $\varepsilon > 0$ is given, there exists a number $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $0 < \text{dist}(\,(x, y), (a, b)\,) < \delta$.

The Geometric Formulation:

Whatever positive ε may be given, when the horizontal planes $z = L + \varepsilon$ and $z = L - \varepsilon$ are drawn:



then it must be possible to find a number $\delta > 0$ such that between these planes you will find all the points of the graph which are vertically aligned with the disk in the xy-plane having center (a, b) and radius δ.

Remark. In the graph above we are considering the limit of the function $z = f(x, y)$ as (x, y) approaches $(0, 0)$. The limit L appears to be 1. The ε given appears to be approximately 0.3. If $L = 1$, then we should be able to find a $\delta > 0$ such that all points (x_0, y_0) within the circle of radius δ centered at $(0, 0)$ will have a corresponding z-value $f(x_0, y_0)$ that falls between the two planes $z = L + \varepsilon$ and $z = L - \varepsilon$.

Which δ appears to work in the surface pictured above for the given epsilon?

Exercise: Use the formal definition to rigorously prove that the limit in the previous example is 0.

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