Maple Lab : Taylor Polynomials (adapted from the Taylor series lab at Colorado State University)

Purpose: The purpose of this lab is to use Maple to compute Taylor polynomials. This allows you to consider Taylor polynomials of much higher order than you could ever conceive of doing by hand. In particular, we are interested in answering the following questions:
• What happens to the Taylor polynomial when its order increases?
• For which values of x is $T_n(x)$ close to $f(x)$?
• How do we measure the difference between $T_n(x)$ and $f(x)$?

Topic 1: TaylorApproximationTutor

Maple has interesting functionality that computes and plots a function and its Taylor polynomial. For this we need to load a package that provides extra functionality by calling

```
> with(Student[Calculus1]) :
```

Now bring up a pop-up window by typing

```
> TaylorApproximationTutor( );
```

in Maple's command window. The window contains a region where the graph of f (in red) is plotted with the graph of the Taylor polynomial $T_n(x)$ (in blue). You can:
• type a new function inside the box next to $f(x) =$,
• choose the point x_0 where the Taylor polynomial is centered using the $x =$ box,
• choose the *Order n* of the Taylor polynomial,
• get a window in which you can choose the plot region using the *Plot Options* button.

Let's experiment with this tool. Compare the function e^{-x^2} centered at $x_0 = 0$, with its Taylor polynomials of order 0, 1, 2, ..., 14 over the interval $[-2, 2]$ with range $[0, 2]$. Remember to type

```
> exp(-x^2)
```

for e^{-x^2} in the $f(x) =$ box.

By changing the order, you can see that the graph of the Taylor polynomial $T_n(x)$ appears to approach the graph of $f(x)$. In fact, according to the TaylorTutor, for values of x near to $x_0 = 0$, the graphs look virtually identical. Notice that the bottom row of the window gives you a Maple command you could use to create the same plot. Also, when you press the *Close* button, the current plot is returned to the Maple worksheet and can be manipulated there.

Question 1:

a) Examine the Taylor polynomials of $\cos(x)$ centered at $x_0 = 0$ over the interval $[-2\pi, 2\pi]$. What happens as the order n increases? When do the graphs appear to become identical?
b) Why is the Taylor polynomial of order 2 equal to the Taylor polynomial of order 3?
c) Start over with Taylor polynomials for $f(x) = \cos(x)$ centered at $x_0 = 2$ and an interval of width 4π centered at x_0, i.e. $[2 - 2\pi, 2 + 2\pi]$. When do the graphs seem to be identical?
d) If you consider the larger interval $[2 - 3\pi, 2 + 3\pi]$, and plot Taylor polynomials centered at $x_0 = 2$, for what order Taylor polynomials do the graphs seem to be identical?
e) In words, describe how the graph of the Taylor polynomial is approaching the graph of f.
f) Give a geometric interpretation (i.e. what type of function are you graphing) of

$T_1(x)$ when $x_0 = 0$ and relate it to the graph of $\cos(x)$.

Topic 2: Computing Error

In the previous exercises, we claimed that f was close to T_n when the graphs overlapped. We'll try to be more precise about the closeness between these two graphs. We know how to measure the distance between the values of f and T_n at a point x. This distance is

$|f(x) - T_n(x)|$.

On the other hand, f and T_n are defined over an interval $[a, b]$. For some choices of x, this distance may be large while for some other values of x, this distance may be small. What do we do when we have to consider a whole range of values for x? We take the largest one!

Definition: The *error* between f and its n-th order Taylor polynomial over the interval $[a, b]$ is the maximum value of

$|f(x) - T_n(x)|$,

when x belongs to $[a, b]$.

In practice, the quantity $|f(x) - T_n(x)|$ is a function of x, and by plotting it, we can estimate its largest value. Let's do this for $\cos(x)$ and its Taylor polynomial of order 4. Type the following.

```
> f := cos(x);
> t4 := TaylorApproximation(f, x, order = 4);
```

This computes a Taylor polynomial approximation of f.

Now we can plot the function $|f(x) - T_4(x)|$ over the interval $[-2, 2]$.

```
> plot(abs(f - t4), x = -2 ..2);
```

We can see that the graph of the function $|f(x) - T_4(x)|$ is always less than 0.09 on the interval $[-2, 2]$ since the range of this interval is less than 0.09. *Therefore we claim that the error between f and T_4 on the interval* $[-2, 2]$ *is less than* 0.09.

If we rescale the y-axis and plot on the interval $[-1, 1]$, we see that the error is smaller (less than 0.002).

```
> plot(abs(f - t4), x = -1 ..1, y = 0 ..0.01);
```

Using *TaylorApproximationTutor*, you could check that the graphs of f and T_4 over the interval $[-2, 2]$ are very close and therefore that the error between f and T_4 should be small.

Question 2: In this question, we will look at the distance between $f(x) = \frac{1}{1 + x^2}$ and its Taylor polynomials centered at 0.

a) Use the *TaylorApproximation* command to find the Taylor polynomials of order 1, 2, 3, 4, and 5.

b) Plot f over the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. On the same plot, draw the Taylor polynomials of order 2 and 4 in different colors. You might want to use Maple's color command to get colors that are easily seen.

c) Over the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, measure the ***error between f and T_n*** for $n = 5$, 10, 20, and 40 by doing the following. Graph each of the four functions $|f(x) - T_n(x)|$, and use the graphs to estimate the error for each n.

d) Repeat question (c) but over the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

e) Graph f together with the Taylor polynomial of order n for $n = 20, 40, 60, 80, 100, 200, 600$. What do you notice happens to the Taylor polynomial approximation as n increases?

Topic 3: An Interesting Function and its Taylor Polynomials

We will now study a very famous function. We won't give away the punch line immediately, so please bear with us. Consider the function $g(x) = e^{-\frac{1}{x^2}}$. This function is not defined for $x = 0$ but can be made continuous by setting it equal to $\lim_{x\to 0} g(x)$ at 0. Since $\lim_{x\to 0} g(x) = 0$, we define the function f by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x < 0 \textbf{ or } 0 < x \\ 0 & x = 0 \end{cases}.$$

We define this function in Maple using a piecewise definition:

> $f := piecewise\left(x < 0 \textbf{ or } x > 0, \exp\left(-\frac{1}{x^2}\right), x = 0, 0\right);$

Plot this function over the interval $[-5, 5]$.

You should see that setting $f(0) = 0$ was perfectly justified. Notice also how flat the graph of f is near $x = 0$.

Now try graphing it over the interval $[-1, 1]$. Because the graph of f is so flat near $x = 0$, we expect that its derivative (and maybe all or a bunch of its derivatives) to be zero at $x = 0$. This is what we will check.

Although the formula for f is not defined when $x = 0$, it is natural to define $f^{(n)}(0)$ to be $\lim_{x\to 0} f^{(n)}(x)$, and this is what Maple does.

Calculate the first derivative:

> $diff(f, x);$

You see that the derivative also has value 0 at 0. Try to evaluate it using

> $subs(x = 0, diff(f, x));$

Maple returns an error when you do so (because of the piecewise definition). But

> $limit(diff(f, x), x = 0);$

will give you the value 0.

For the same reasons as before, the derivative is now defined for all x and is continuous. We can therefore compute its Taylor polynomial centered at $x_0 = 0$. The command

> $TaylorApproximation(f, x, order = 3);$

will give you an error for similar reasons as above. We are going to have to compute the Taylor polynomial the hard way, by evaluating derivatives (using *limit*).

Question 3:

a) Compute the value of $f^{(n)}(0)$ for $n = 3$, 4, and 5. (Remember, when we refer to f here, we refer to the f whose value at zero is defined to be the limit of f as x approaches zero. Also recal that you can compute the n-th derivative by using *diff*$(f, x\$n)$.) Make a guess for the value of $f^{(100)}(0)$ and check it.

b) Using your guess from (a), compute the Taylor polynomial for f centered at 0 of order 100.

c) Compute the error between f and T_n over the interval $[-1, 1]$ for $n = 1$, 2, 3, 4, and 5.

d) Do the Taylor polynomials approach f as n increases?

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