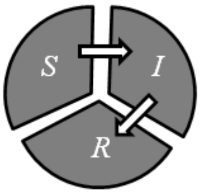
Spread of Disease—The *SIR* model[1]

Many common childhood diseases (for instance, the measles) have a common sort of structure. They divide the population into three separate groups: those that are susceptible to the disease because they haven't had it, those that are actually sick, and those that are recovered and are subsequently immune to the disease. We denote by S the number of susceptible individuals, by I the number of infected individuals, and by R the number of recovered individuals. In this project, you will explore a mathematical model (called the *SIR* model) that attempts to capture this structure, and you will use it to study an epidemic.

Constructing the Model: Members of the population move from one subpopulation to another according to the following diagram:



Members of the susceptible population acquire the disease and move into the infected population. After a period of time, those that are sick get well and move into the recovered population. Those that are recovered, stay recovered. (We are making some simplifying assumptions: This assumes a disease with no mortality rate and one for which immunity is permanent. Moreover, it assumes a fairly short-lived epidemic so one need not consider immigration or emigration in the population. Other diseases require a more complex model.)

In order to set up our differential equations model, we will need to describe the rates of change of S, I, and R.

Question 1: S changes when one or more members of the susceptible population become infected. This requires interaction between members of the susceptible population and members of the infected population. Write a differential equation that describes the S' in terms of S and I. (**Note**: what can you say about the sign of S'? Your differential equation will need to take this into account.)

Question 2: Suppose that the disease in question is relatively homogeneous in that the disease runs its course in pretty much the same length of time for everyone who gets it. Suppose that it takes on average d days to go from infection to recovery. Write a differential equation that describes the rate of change of the recovered population. (You will need to make a simplifying assumption.)

Question 3: I changes either when someone gets sick or when someone gets well. To be specific, I increases when someone gets sick, and decreases when someone gets well. Write a differential equation that describes the rate of change of the infected population.

Analysis of the Model: We will initially consider a measles epidemic. Assume that when we begin to study the epidemic on a Wednesday, when we have 45,400 susceptible individuals, 2100 infected

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[1] This lab is adapted from the book <u>Calculus in Context</u> by Callahan et. al. Some portions of the lab are taken verbatim from that source.

individuals, and 2500 recovered individuals. The measles last about 14 days and .00001 is a reasonable choice for the transmission rate.

Question 4: Analyze the model using Euler's method in order to answer the following questions:

a) How many susceptible, infected, and recovered people will there be ten days into the epidemic?

b) How many were in each population on the previous Sunday?

c) How long will it take for the epidemic to die out?

Question 5: There are a number of pieces of information that we can get without fully analyzing the model. (That is, without using Euler's method to solve the system of equations.)

a) When will the epidemic peak? Think about what this tells you about I' and use calculus to solve analytically for the exact number of people who will be sick at the peak. Approximately *when* does the epidemic peak?

b) **Threshold**: Epidemics are more likely to occur in situations where there are large numbers of people. This is partially because crowding enhances the chances that susceptible and infected peoples will come into contact, but it is also an artifact of the disease itself. In fact, there must be a certain number of susceptibles in the population, or the epidemic will never take hold---to be specific, unless the number of susceptibles is greater than the threshold for the disease, the number of infected will simply decrease. Find the threshold for the measles epidemic described by our model.

Question 6: Using Euler's method, get a graph that shows the progression of the functions S, I, and R. Does your graph correctly reflect the answers to the questions that you have already answered in other ways?

Question 7---Quarantine: One of the ways to treat an epidemic is to keep those that are infected away from those that are susceptible; this is called *quarantine*. The intention is to reduce the chance that the illness will be transmitted to a susceptible person. Thus, quarantine alters the *transmission coefficient*.

a) Suppose a quarantine is put into effect that cuts in half the chance that a susceptible will fall ill. What is the new transmission coefficient?

b) Use Euler's method to find out how long it will take for the epidemic to die out in the presence of this quarantine. Get both a graph that shows this time-frame and a numerical answer.

c) What is the new threshold for the disease? Is the quarantine sufficiently drastic to eliminate the epidemic? If not, find the largest value that the transmission coefficient can have and still guarantee that I never increases. What level of quarantine does this represent? That is, do you have to reduce the chance that a susceptible will fall ill to one-third of what it was with no quarantine at all? To one-fourth, or what?