Lab II: Riemann Sums

*Name:*________________________ *Date:*________________________

A Riemann sum S_n for the function $f(x)$ is defined to be $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x$. In these sums, n is the number of subintervals into which the interval $[a, b]$ is divided by equally spaced partition points a = $x_0 < x_1 < \ldots < x_{n-1} < x_n$ = b. The length of each subinterval is $\Delta x = \dfrac{b-a}{n}$. (The subintervals don't have to be all the same length, but calculations are easier when they are.) Each c_k is a number from the k^{th} subinterval. We typically use a rule for choosing the c_k's, such as always choosing c_k to be the left endpoint of the subinterval, but it is not necessary to use a rule at all.

The ***partition points*** $x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n$ that divide the interval of integration $a \le x \le b$ into subintervals are at the heart of the idea of a Riemann sum. But the ***evaluation points*** $c_1, c_2, c_3, c_4, \ldots, c_{n-1}, c_n$ are more prominent in calculating the value of a Riemann sum.

1. We begin with a left Riemann sum with 6 subdivisions of equal length as an approximation of the integral $\int_{-1}^{2} \sqrt{x^3+1}\, dx$. On the graph of $f(x) = \sqrt{x^3+1}$ (next page) , mark and label the points $x_0, x_1, x_2, x_3, x_4, x_5, x_6$ that divide the interval [-1, 2] into six subintervals of equal length. The seven endpoints of these six subintervals are the ***partition points***.

   We'll first calculate a Riemann sum using the left endpoints of the subintervals as the evaluation points. With this choice of evaluation points, $c_1, c_2, c_3, c_4, c_5, c_6$ coincide with the first six partition points $x_0, x_1, x_2, x_3, x_4, x_5$. Label $c_1, c_2, c_3, c_4, c_5, c_6$ on the graph of $f(x)$.

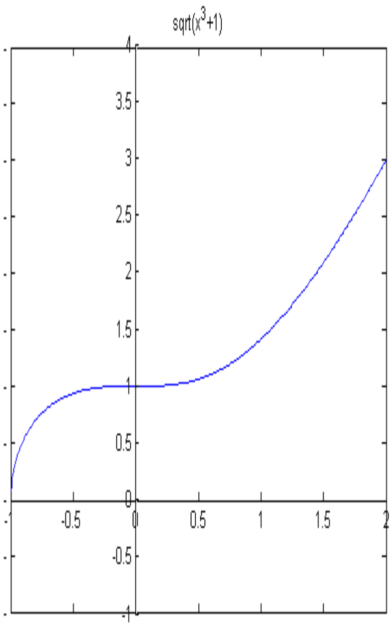
   Write a list of the 6 values that determine the heights of your rectangles. Then draw the rectangles on the graph.

   Now calculate the approximation to the integral by adding the areas of the six rectangles (round to 6 decimal places).

   Write a formula using sigma notation for the sum you just computed in terms of k (your answer should be specific enough that it doesn't have a c_k in it). Then implement your formula in Maple. (You will want to use the sum command).

$$L_6 = \Sigma$$

$$=$$



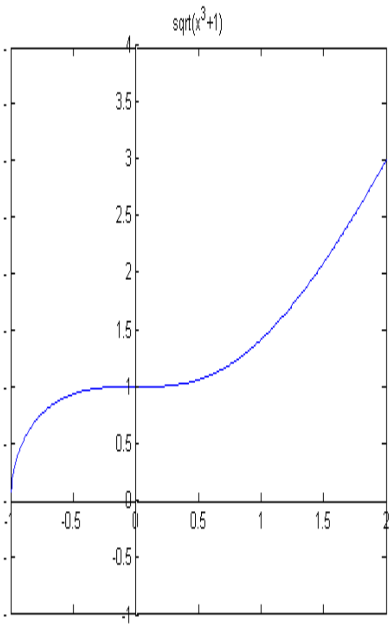
2. The approximation for $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ calculated in #1 is not very accurate. Use the graph of $f(x) = \sqrt{x^3+1}$ above to explain why L_6 is too small.

Will a Riemann sum approximation for the definite integral $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ calculated using left endpoints as evaluation points always be too small? Use graphs and words to explain why or why not.

3. Next, calculate the Riemann sum R_6 for the integral $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ using the right endpoints of the subintervals as evaluation points.

The partition points are the same: -1, -0.5, 0, 0.5, 1, 1.5, 2. The evaluation points, however, are different. Label $x_0, x_1, x_2, x_3, x_4, x_5, x_6$ and $c_1, c_2, c_3, c_4, c_5, c_6$ on the graph of $f(x)$ below.

Compute R_6 and illustrate (label!) all steps of this calculation on the graph of $f(x) = \sqrt{x^3+1}$ below.



Write a formula using sigma notation for the sum you computed in terms of k.

$$R_6 = \Sigma$$

$$=$$

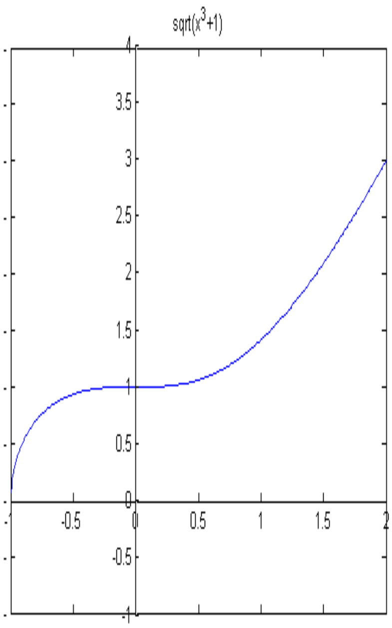
4 The approximation for $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ using right endpoint evaluations is not very accurate. Use the graph of $f(x) = \sqrt{x^3+1}$ to explain why R_6 is too large.

Will a Riemann sum approximation for the definite integral $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ calculated using right endpoints as evaluation points always be too large? Use graphs and words to explain why or why not.

5. Calculate the Riemann sum M_6 for the integral $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ using midpoints of the subintervals as evaluation points.

The partition points are the same: -1, -0.5, 0, 0.5, 1, 1.5, 2. The evaluation points, however, are different. Label $x_0, x_1, x_2, x_3, x_4, x_5, x_6$ and $c_1,\ c_2,\ c_3,\ c_4, c_5,\ c_6$ on the graph of $f(x)$ below.

Compute M_6 and illustrate (label!) all steps of this calculation on the graph of $f(x) = \sqrt{x^3+1}$ below.



Write a formula using sigma notation for the sum you computed in terms of k.

$$M_6 = \Sigma$$

$$=$$

6. Use the graphs of $f(x) = \sqrt{x^3+1}$ to explain why M_6 is more accurate than the approximations to the integral calculated previously.

7. Do you think the approximation to the integral calculated with midpoint evaluations is larger or smaller than the actual value of the integral? Explain why you think so.

8. Use Maple to evaluate $\int_{-1}^{2} \sqrt{x^3+1}\, dx$ numerically and record the result. ___________________

   Will choosing midpoints as evaluation give the best approximation for **every** definite integral $\int_{-1}^{2} f(x)dx$ (no matter what continuous function $f(x)$ is the integrand)? Use examples to explain why or why not.