Math 224
Vector Spaces

Definition. A **vector space** a set V of objects called **vectors**, together with :

- A rule for adding any two vectors $\mathbf{v}$ and $\mathbf{w}$ to produce a vector $\mathbf{v}+\mathbf{w}$ in V. Note that this means that V must be **closed under vector addition**.
- A rule for multiplying any vector $\mathbf{v}$ in V by any scalar r in $\mathbb{R}$ to produce a vector $r\mathbf{v}$ in V. Note that this means that V must be **closed under scalar multiplication**.
- There must exist a vector $\mathbf{0}$ in V. Note: this may not always be the vector that you expect to call the zero vector. For each vector $\mathbf{v}$ in V, there must exist a vector $-\mathbf{v}$ in V (see below).
- Properties A1 through A4 and S1 through S4 below must be satisfied for all choices of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars r, s in $\mathbb{R}$.

  A1 $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$. This means that vector addition must be associative.

  A2 $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$. This means that vector addition must be commutative.

  A3 $\mathbf{0}+\mathbf{v} = \mathbf{v}$. This means that $\mathbf{0}$ must the the additive identity of the vector space.

  A4 $\mathbf{v}+-\mathbf{v} = \mathbf{0}$. This means that $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$.

  S1 $r(\mathbf{v}+\mathbf{w}) = r\mathbf{v}+r\mathbf{w}$). This is a distributive rule for scalar multiplication.

  S2 $(r+s)\mathbf{v} = r\mathbf{v}+s\mathbf{v}$. This is another distributive rule for scalar multiplication.

  S3 $r(s\mathbf{v}) = (rs)\mathbf{v}$. This means that scalar multiplication is associative.

  S4 $1\mathbf{v} = \mathbf{v}$. This means that scalar multiplication preserves scale.

Examples of vector spaces.

1. $\mathbb{R}^n$, with usual vector addition and scalar multiplication, is a vector space for every integer $n \geq 1$.
2. Let F be the set of all real-valued functions with domain $\mathbb{R}$. The vector sum of two functions f and g in F is defined in the usual way by $(f+g)(x) = f(x)+g(x)$. For any scalar r in $\mathbb{R}$, scalar multiplication is defined by $(rf)(x) = rf(x)$ is a vector space. Show that F with these operations is a vector space.
3. Show that the set P of all polynomials in the variable x with coefficients in $\mathbb{R}$, with vector addition and scalar multiplication the usual addition of polynomials and multiplication of a polynomial by a scalar, is a vector space.

4. Show that the set $M_{m,n}$ of all $m \times n$ matrices with real entries, with the usual addition of matrices and multiplication of a matrix by a scalar, is a vector space.

5. Show that the set P_n of all polynomials in x with coefficients in $\mathbb{R}$ of degree less than or equal to n, together with the zero polynomial, with usual addition and scalar multiplication, is a vector space.

6. Consider the set $\mathbb{R}^2$, with the usual addition, but with scalar multiplication defined by $r[x, y] = [ry, rx]$. Show that this set is a vector space.

7. Consider the set $\mathbb{R}^2$, with addition defined by $[x, y] \ddagger [a, b] = [x + a + 1, y + b]$, and with scalar multiplication defined by $r[x, y] = [rx + r - 1, ry]$. Is this set a vector space? Show, by verifying A1-A4 and S1-S4, that this set is a vector space. Note: The zero vector will NOT be the vector $[0, 0]$. What is the zero vector in this vector space.

8. Determine whether the given set is closed under the usual operations of addition and scalar multiplication, and is a vector space.

   (a) The set of all upper-triangular $n \times n$ matrices (i.e. the set of all $n \times n$ matrices with zeros below the main diagonal).

   (b) The set of all 2×2 matrices of the form
   $$\begin{bmatrix} r & 1 \\ 1 & s \end{bmatrix},$$
   where r and s are any real numbers.

   (c) The set of all diagonal $n \times n$ matrices.

   (d) The set of all 3×3 matrices of the form
   $$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix},$$
   where a, b, c, d, e are any real numbers.

   (e) The set $\{0\}$ consisting of only the number 0.

   (f) The set $\mathbb{Q}$ of rational numbers.

   (g) The set $\mathbb{C}$ of complex numbers:
   $$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\},$$
   where $i = \sqrt{-1}$.