Math 224
Tuesday, November 27, 2007
Basic Properties of Vector Spaces

1. **Definition.** Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ in a vector space V and scalars $r_1, r_2, \ldots, r_k$ in $\mathbb{R}$, the vector

$$r_1\mathbf{v_1} + r_2\mathbf{v_2} + \ldots + r_k\mathbf{v_k}$$

is a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ with coefficients $r_1, r_2, \ldots, r_k$.

Example.

2. **Definition.** Let X be a subset of a vector space V. The **span** of X is the set of all linear combinations of vectors in X, and is denoted $\mathrm{sp}(X)$.

Example.

3. **Definition.** A subset W of a vector space V is a **subspace** of V if W is nonempty and satisfies the following two conditions:

   (a) W is closed under vector addition: if $\mathbf{v}$ and $\mathbf{w}$ are in W, then $\mathbf{v} + \mathbf{w}$ is in W.

   (b) W is closed under scalar multiplication: if $\mathbf{v}$ is in W and r is any scalar in $\mathbb{R}$, then $r\mathbf{v}$ is in W.

   Example.

4. **Definition.** Let X be a set of vectors in a vector space V. A **dependence relation** among vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ in the set X is an equation of the form

$$r_1\mathbf{v_1} + r_2\mathbf{v_2} + \ldots + r_k\mathbf{v_k} = \mathbf{0},$$

where at least one of the scalars r_j is NOT equal to 0. If such a dependence relation exists, then X is a **linearly dependent** set of vectors. If no such relation exists, then X is a **linearly independent** set of vectors.

Example.

5. **Definition.** Let V be a vector space. A set of vectors in V is a **basis** for V if the following two conditions are met:

   (a) The set of vectors spans V, i.e. any vector in V can be written as a linear combination of vectors in the basis set.

   (b) The set of vectors is linearly independent.

   Example.

6. **Definition.** The number of elements in a basis for a (finitely generated) vector space V is the **dimension** of V, and is denoted by $\dim(V)$.

   Example.