Math 224
Quiz 7 Solutions
Thursday, November 29, 2007

Note: You are allowed to use Maple for this quiz, but you must show all work to receive credit.

1. Is the matrix
$$A = \begin{bmatrix} 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & 2/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$$
orthogonal? Why or why not?

   Solution. A is orthogonal since $A^T A = I$.

2. Let $D = C^{-1}AC$ be a diagonal matrix, where C is an orthogonal matrix. Show that A is symmetric.

   Solution. Since C is orthogonal, $C^{-1} = C^T$. We can rewrite $C^{-1}AC = D$ as $A = CDC^{-1}$. Thus:
$$\begin{aligned} A^T &= (CDC^{-1})^T \\ &= (CDC^T)^T \\ &= (C^T)^T D^T C^T \\ &= CDC^T \\ &= CDC^{-1} \\ &= A \end{aligned}$$

3. Find the projection matrix for the subspace
$$W = \mathrm{sp}([1,2,1,1], [-1,1,0,-1])$$
in $\mathbb{R}^4$, and use it to find the projection of $\mathbf{b} = [1,2,1,3]$ on W.

   Solution. First, we form the matrix
$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}.$$
   Then the projection matrix is
$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 10/21 & -1/21 & 3/21 & 10/21 \\ -1/21 & 19/21 & 6/21 & -1/21 \\ 3/21 & 6/21 & 3/21 & 3/21 \\ 10/21 & -1/21 & 3/21 & 10/21 \end{bmatrix}.$$

Thus the projection of $\mathbf{b}$ on W is

$$\mathbf{b}_W = P\mathbf{b} = [41/21, 40/21, 9/7, 41/21].$$

4. Show that the projection matrix

$$P = A(A^TA)^{-1}A^T$$

satisfies $P^2 = P$. (Here A is the $n \times k$ matrix with column vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}$, where $\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}\}$ is a basis for a subspace W of $\mathbb{R}^n$).

Solution.

$$\begin{aligned} P^2 &= (A(A^TA)^{-1}A^T)^2 \\ &= (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) \\ &= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= P \end{aligned}$$