Math 224
Quiz 6 Solutions
Thursday, November 15, 2007

Note: You are allowed to use Maple for computation of dot products, addition of vectors, row reduction, etc. on this quiz, but you must SHOW ALL WORK to receive credit. In particular, you may not use the GramSchmidt command in Maple to produce an orthogonal or orthonormal basis (you may, however, use it to check your work). If you need more space for your work, use the back of the pages and/or attach additional sheets.

1. Find the projection of $[-1,2,0,1]$ on $sp([2,-3,1,2])$ in $\mathbb{R}^4$.

   Solution. Recall that the projection of $\mathbf{b}$ on $W = sp(\mathbf{a})$ is given by
   $$\mathbf{b_W} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}.$$
   Here we have $\mathbf{b} = [-1,2,0,1]$ and $\mathbf{a} = [2,-3,1,2]$, so
   $$\mathbf{b}\cdot\mathbf{a} = -6 \text{ and } \mathbf{a}\cdot\mathbf{a} = 18.$$
   So
   $$\mathbf{b_W} = \frac{-6}{18}[2,-3,1,2] = [-2/3, 1, -1/3, -2/3].$$

2. Find the orthogonal complement of $W = sp([1,3,0],[2,1,4])$.

   Solution. We find the nullspace of the matrix
   $$A = \left[\begin{array}{ccc} 1 & 3 & 0 \\ 2 & 1 & 4 \end{array}\right].$$
   $$\text{rref}(A|\mathbf{0}) - \left[\begin{array}{ccc|c} 1 & 0 & 12/5 & 0 \\ 0 & 1 & -4/5 & 0 \end{array}\right],$$
   so
   $$W^{\perp} = sp([-12/5, 4/5, 1]) = sp([-12, 4, 5]).$$

3. Find an orthonormal basis for the subspace $sp([0,1,0],[1,1,1])$ of $\mathbb{R}^3$.

   Solution.
   $$\mathbf{v_1} = [0,1,0].$$
   $$\begin{aligned} \mathbf{v_2} &= [1,1,1] - \frac{[1,1,1]\cdot[0,1,0]}{[0,1,0]\cdot[0,1,0]}[0,1,0] \\ &= [1,1,1] - [0,1,0] \\ &= [1,0,1] \end{aligned}$$

Thus an orthogonal basis is

$$\{[0,1,0],[1,0,1]\}.$$

Normalizing, an orthonormal basis is

$$\{[0,1,0],[\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}]\}.$$

4. Let W be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Let $\mathbf{b}$ be any vector in $\mathbb{R}^n$. Writing $\mathbf{b} = \mathbf{b_W} + \mathbf{b_{W^\perp}}$, where $\mathbf{b_W}$ is in W and $\mathbf{b_{W^\perp}}$ is in $W^\perp$, prove that

$$||\mathbf{b}|| = \sqrt{||\mathbf{b_W}||^2 + ||\mathbf{b_{W^\perp}}||^2}.$$

Hint: Use $||\mathbf{b}||^2 = \mathbf{b}\cdot\mathbf{b}$.

Solution.

$$\begin{aligned} ||\mathbf{b}||^2 &= \mathbf{b}\cdot\mathbf{b} \\ &= (\mathbf{b_W} + \mathbf{b_{W^\perp}})\cdot(\mathbf{b_W} + \mathbf{b_{W^\perp}}) \\ &= \mathbf{b_W}\cdot\mathbf{b_W} + \mathbf{b_{W^\perp}}\cdot\mathbf{b_{W^\perp}} + 2\mathbf{b_W}\mathbf{b_{W^\perp}} \\ &= ||\mathbf{b_W}||^2 + ||\mathbf{b_{W^\perp}}||^2 \end{aligned}$$

Thus

$$||\mathbf{b}|| = \sqrt{||\mathbf{b_W}||^2 + ||\mathbf{b_{W^\perp}}||^2}.$$