# Math 224
# Quiz 5
# Thursday, October 25, 2007

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**Note:** You are allowed to use Maple for computation on this quiz, but you must SHOW ALL WORK to receive credit. Answers for which no work is shown will receive NO CREDIT.

1. Consider the matrix
$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$
and the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v_4} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v_5} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. List the vectors that are eigenvectors of $A$, and for the ones that are eigenvectors of $A$, find the corresponding eigenvalue.

   **Solution.** $\mathbf{v_1}$, $\mathbf{v_3}$, and $\mathbf{v_5}$ are eigenvalues of $A$ with corresponding eigenvalues -1, 2, and 2, respectively.

2. Find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors of the matrix
$$A = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}.$$

   **Solution.**
$$p(\lambda) = (\lambda - 1)(\lambda - 3),$$
   so the eigenvalues of $A$ are $\boxed{\lambda_1 = 1 \text{ and } \lambda_2 = 3}$. The eigenvectors corresponding to $\lambda_1 = 1$ are $\boxed{\mathbf{v_1} = \begin{bmatrix} -r \\ r \end{bmatrix},\ r \neq 0}$. The eigenvectors corresponding to $\lambda_2 = 3$ are $\boxed{\mathbf{v_2} = \begin{bmatrix} -\frac{1}{2}s \\ s \end{bmatrix},\ s \neq 0}$.

3. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation, and let $\lambda$ be a scalar. Prove that $F_\lambda = \{\mathbf{v} \in \mathbf{R}^n \mid T(\mathbf{v}) = \lambda\mathbf{v}\}$ is a subspace of $\mathbf{R}^n$.

   **Solution.** First, note that $F_\lambda$ is non-empty since $\mathbf{0}$ is in $F_\lambda$ (since for any linear transformation $T$, we must have $T(\mathbf{0}) = \mathbf{0}$). Next, we must show that $F_\lambda$ is closed under vector addition and scalar multiplication. So, let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $F_\lambda$, and let $\lambda$ be a scalar. Then $T(\mathbf{v}) = \lambda\mathbf{v}$ and $T(\mathbf{w}) = \lambda\mathbf{w}$. First, we check that $F_\lambda$ is closed under vector addition:

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(\mathbf{v}) + T(\mathbf{w}) \\ &= \lambda\mathbf{v} + \lambda\mathbf{w} \\ &= \lambda(\mathbf{v}+\mathbf{w}) \end{aligned}$$

Thus $\mathbf{v}+\mathbf{w}$ is in $F_\lambda$, so $F_\lambda$ is closed under vector addition. Next, we check that $F_\lambda$ is closed under scalar multiplication:

$$\begin{aligned} T(r\mathbf{v}) &= rT(\mathbf{v}) \\ &= r\lambda\mathbf{v} \\ &= \lambda(r\mathbf{v}) \end{aligned}$$

Thus $r\mathbf{v}$ is in $F_\lambda$, so $F_\lambda$ is closed under scalar multiplication. We conclude that $F_\lambda$ is a subspace of $\mathbf{R}^n$.

4. Let $A$ be an $n \times n$ real matrix. Prove that the eigenvalues of $A$ are the same as the eigenvalues of $A^T$.

   **Solution.** It suffices to show that the characteristic polynomial of $A$ is equal to the characteristic polynomial of $A^T$. We have:

$$\begin{aligned} \det(A-\lambda I) &= \det((A-\lambda I)^T) \\ &= \det(A^T - (\lambda I)^T) \\ &= \det(A^T - \lambda I) \end{aligned}$$

Thus the characteristic polynomial of $A$ is equal to the characteristic polynomial of $A^T$, so the eigenvalues of $A$ are the same as the eigenvalues of $A^T$.