Math 224
Tuesday, November 6, 2007
Projections

How to find the orthogonal complement of a subspace W of $\mathbf{R}^n$

1. Find a basis for W. Often, this will be given.
2. Form the matrix A whose *rows* are the basis vectors for W.
3. The nullspace of A (i.e. the solution space of $A\mathbf{x} = \mathbf{0}$) is $W^\perp$.

Exercises.

1. Prove that $W^\perp$ is a subspace of $\mathbf{R}^n$.

2. Suppose that $\dim(W) = k$. Find $\dim(W^\perp)$.

An important property. Any vector $\mathbf{b}$ in $\mathbf{R}^n$ can be expressed uniquely in the form

$$\mathbf{b} = \mathbf{b}_W + \mathbf{b}_{W^\perp}$$

where $\mathbf{b}_W$ is in W and $\mathbf{b}_{W^\perp}$ is in $W^\perp$. We call $\mathbf{b}_W$ the **projection** of $\mathbf{b}$ on W.

Note: The steps that follow are what the textbook refers to as the *boxed procedure* on p. 333. You should use these steps to find the projection of $\mathbf{b}$ on W (and understand why they work).

How to find the projection of b on W.

1. Find a basis $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ for the subspace W. Often, this will be given.
2. Form the matrix A whose row vectors consist of the basis vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ for W.
3. Find a basis $\{\mathbf{v_{k+1}}, \mathbf{v_{k+2}}, \ldots, \mathbf{v_n}\}$ for $W^\perp$ by solving $A\mathbf{x} = \mathbf{0}$.
4. Then $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for $\mathbf{R}^n$.
5. Write the original vector $\mathbf{b}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. This means that we need to find scalars $r_1, r_2, \ldots, r_n$ such that

   $$\mathbf{b} = r_1\mathbf{v_1} + r_2\mathbf{v_2} + \ldots r_n\mathbf{v_n}.$$

   Recall that we can do this by solving the linear system $M\mathbf{r} = \mathbf{b}$, where M is the matrix whose column vectors consist of the basis vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. To solve the linear system, we row-reduce the augmented matrix $[M|\mathbf{b}]$ and find $r_1, r_2, \ldots, r_n$.
6. Then $\mathbf{b}_W = r_1\mathbf{v_1} + r_2\mathbf{v_2} + \ldots + r_k\mathbf{v_k}$ is the projection of $\mathbf{b}$ on W.