Math 224
Thursday, November 15, 2007

$\mathbf{rank}(A^TA) = \mathbf{rank}(A)$

Let A be an $m \times n$ matrix. Then A^TA is an $n \times n$ matrix with the **same rank** as A.

Proof.

1. How many columns does A have? What is the size of A^TA? How many columns does A^TA have?

2. Write down the rank equation for the matrix A.

3. Write down the rank equation for the matrix A^TA.

4. Conclude that if we can show that nullity(A)=nullity(A^TA), then we can conclude that A and A^TA have the same rank.

5. Our goal now is to show that A and A^TA have the same nullspace.

   (a) Show that if $\mathbf{v}$ is a vector in the nullspace of A, then $\mathbf{v}$ must also be in the nullspace of A^TA.

(b) Show if that if $\mathbf{v}$ is a vector in the nullspace of $A^T A$, then $\mathbf{v}$ must also be in the nullspace of A. Be careful: this is tricky. Hint: try to show that $||A\mathbf{v}|| = 0$, and conclude that $A\mathbf{v} = \mathbf{0}$.

(c) Conclude that A and $A^T A$ have the same nullspace.

6. Conclude that A and $A^T A$ have the same rank.