Math 224
Practice Exam 3

Note: You are allowed to use Maple, but you must show all work to receive credit. In particular, you are allowed to use the GramSchmidt command in Maple.

1. (a) Suppose that A is an $n \times 3$ matrix with columns $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Suppose also that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are mutually orthogonal and that $||\mathbf{v_1}|| = ||\mathbf{v_2}|| = ||\mathbf{v_3}|| = 4$. Find $A^T A$.
   (b) Suppose that A is an $n \times 3$ matrix with columns $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Suppose also that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are mutually orthogonal and that $||\mathbf{v_1}|| = 1, ||\mathbf{v_2}|| = 2, ||\mathbf{v_3}|| = 3$. Find $A^T A$.

2. Consider the subspace
$$W = \text{sp}([1,2,3,4],[5,6,7,8])$$
of $\mathbb{R}^4$.
   (a) Find a basis for $W^\perp$.
   (b) Write $\mathbf{b} = [3, -2, 1, 5]$ in the form
$$\mathbf{b} = \mathbf{b_W} + \mathbf{b_{W^\perp}},$$
   where $\mathbf{b_W}$ is in W and $\mathbf{b_{W^\perp}}$ is in $W^\perp$.
   (c) Find the projection matrix P for W.
   (d) Find $\mathbf{b_W}$ using P, and confirm that you get the same result using both methods.

3. (a) Show that the set $\{[2,3,1],[-1,1,-1]\}$ is orthogonal.
   (b) Find the projection of $\mathbf{b} = [2,1,4]$ on $W = \text{sp}([2,3,1],[-1,1,-1])$. Hint: you can make the computation easier by using your result from part (a).

4. Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $[1,1,1]$.

5. If A is an orthogonal matrix, show that $||A\mathbf{x}|| = ||\mathbf{x}||$.

6. What are the possible eigenvalues of a projection matrix?

7. Suppose that P is the projection matrix corresponding to a 3-dimensional subspace W of $\mathbb{R}^4$. What is the rank of P?

8. Explain geometrically why $P^2 = P$ for any projection matrix P.

9. Suppose that $\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}\}$ is an *orthonormal* basis for a subspace W of $\mathbb{R}^n$. Show that the projection matrix for W is $P = AA^T$.

10. Find the coordinate vector of $x + x^4$ in P_4 relative to the basis $B = (1, 2x - 1, x^3 + x^4, 2x^3, x^2 + 2)$.

11. Show that the set $B' = ((x+1)^3, (x+1)^2, x+1, 1)$ is a basis for P_3.

12. Determine whether each of the following statements are True or False. No explanation is necessary.

    (a) Every vector space contains at least one vector.

    (b) Every vector space contains at least two vectors.

    (c) Any two bases in a finite-dimensional vector space V have the same number of elements.

    (d) The projection of $\mathbf{b}$ on $\mathrm{sp}(\mathbf{a})$ is a scalar multiple of $\mathbf{b}$.

    (e) The projection of $\mathbf{b}$ on $\mathrm{sp}(\mathbf{a})$ is a scalar multiple of $\mathbf{a}$.

    (f) The intersection of W and $W^\perp$ is empty.

    (g) If A and B are orthogonal matrices, then AB is orthogonal.

    (h) Every projection matrix is symmetric.