Math 224
Practice Exam 2

1. Find each of the following, given that A is a 3×3 matrix with determinant 7.
   (a) $\det(3A)$
   (b) $\det(A^{-1})$
   (c) $\det(2A^{-1})$
   (d) $\det((2A)^{-1})$
2. Show that a square matrix A is invertible if and only if $A^T A$ is invertible.
3. Find the volume of the 4-box in $\mathbf{R}^4$ determined by the vectors
$$[1,-1,0,1],\ [2,-1,3,1],\ [-1,4,2,-1],\ [0,1,0,2].$$
4. Prove the following identity for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and scalars k:
$$(\mathbf{u}+k\mathbf{v})\times\mathbf{v} = \mathbf{u}\times\mathbf{v}$$
5. Diagonalize the following matrix A, if possible:
$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$
6. Diagonalize the following matrix A, and find a formula for A^k in terms of k:
$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$
7. Suppose that the characteristic polynomial of A is
$$p(\lambda) = (\lambda-1)(\lambda-3)^2(\lambda-4)^3.$$
   What is the size of A? Is A invertible? Explain your reasoning.
8. Show that if $\mathbf{x}$ is an eigenvector of AB, and if $B\mathbf{x} \neq 0$, then $B\mathbf{x}$ is an eigenvector of BA.
9. Show that similar matrices have the same eigenvalues.

10. Determine whether each of the following statements are True or False. No explanation is necessary.

    (a) For any $n \times n$ matrix A, $\det(4A) = 4\det(A)$.

    (b) For any $n \times n$ matrix A, $\det(I_n + A) = 1 + \det(A)$, where I_n is the $n \times n$ identity matrix.

    (c) There is no square matrix A such that $\det(A^T A) = -1$.

    (d) Each eigenvalue of A is also an eigenvalue of A^2.

    (e) Eigenvalues must be non-zero scalars.

    (f) Eigenvectors must be non-zero vectors.

    (g) If A is an $n \times n$ diagonalizable matrix, then each vector in $\mathbf{R}^n$ can be written as a linear combination of eigenvectors of A.

    (h) If a 5×5 matrix A has fewer than 5 distinct eigenvalues, then A is not diagonalizable.