Math 224

Practice Exam 3.

Solutions.

$\boxed{\text{Problem 1}}$ (a) $A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}_{n\times 3}$

$$A^T = \begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ — & v_3 & — \end{bmatrix}_{3\times n}$$

$$A^T A = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_3 \cdot v_1 & v_3 \cdot v_2 & v_3 \cdot v_3 \end{bmatrix}_{3\times 3} = \begin{bmatrix} \|v_1\|^2 & 0 & 0 \\ 0 & \|v_2\|^2 & 0 \\ 0 & 0 & \|v_3\|^2 \end{bmatrix}$$

Thus $A^T A = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} = 16 I_3$.

(b) Using the same reasoning as in part (a),

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

$\boxed{\text{Problem 2}}$ $W = sp([1,2,3,4], [5,6,7,8])$.

(a) To find a basis for $W^\perp$, we find a basis for the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\mathrm{rref}(A) = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \end{array}$$

let $x_3 = r$, $x_4 = s$.

$x_1 = r + 2s$

$x_2 = -2r - 3s$.

$$\vec{x} = r \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow W^{\perp} = \text{nullspace}(A)$

$= sp([1,-2,1,0], [2,-3,0,1])$.

(b). To write $\vec{b} = [3,-2,1,5]$ in the form $\vec{b} = \vec{b_W} + \vec{b_{W^\perp}}$,

we need to find $r_1, r_2, r_3, r_4$ such that

$$\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix} = r_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + r_2 \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus we form the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 5 & 1 & 2 & 3 \\ 2 & 6 & -2 & -3 & -2 \\ 3 & 7 & 1 & 0 & 1 \\ 4 & 8 & 0 & 1 & 5 \end{array}\right]$$

and row-reduce:

$$\left[\begin{array}{c|c} & 41/40 \\ I_4 & -1/8 \\ & -6/5 \\ & 19/10 \end{array}\right]$$

$$\Rightarrow \vec{b}_W = \frac{41}{40}[1,2,3,4] - \frac{1}{8}[5,6,7,8]$$

$$= \left[\frac{2}{5}, \frac{13}{10}, \frac{11}{5}, \frac{31}{10}\right]$$

$$\vec{b}_{W^\perp} = -\frac{6}{5}[1,-2,1,0] + \frac{19}{10}[2,-3,0,1]$$

$$= \left[\frac{13}{5}, \frac{-33}{10}, -\frac{6}{5}, \frac{19}{10}\right]$$

(c) let

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

$$P = A\,(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 7/10 & 2/5 & 1/10 & -1/5 \\ 2/5 & 3/10 & 1/5 & 1/10 \\ 1/10 & 1/5 & 3/10 & 2/5 \\ -1/5 & 1/10 & 2/5 & 7/10 \end{bmatrix}$$

(d) $\vec{b}_W = P\vec{b}$

$$= [2/5,\ 13/10,\ 11/5,\ 31/10]\ \checkmark$$

$\boxed{\text{Problem 3}}$ (a) $[2,3,1]\cdot[-1,1,-1] = -2+3-1$

$= 0,$

so the set $\{[2,3,1],[-1,1,-1]\}$ is orthogonal.

(b) $\vec{b} = [2,1,4]$ $W = sp([2,3,1],[-1,1,-1])$.

Since $\vec{a}_1 = [2,3,1]$ and $\vec{a}_2 = [-1,1,-1]$ are orthogonal, the projection of $\vec{b}$ on $sp(\vec{a}_1,\vec{a}_2)$ is:

$$\vec{b}_W = \frac{b\cdot a_1}{a_1\cdot a_1}a_1 + \frac{b\cdot a_2}{a_2\cdot a_2}a_2$$

$$= \frac{11}{14}[2,3,1] + -\frac{5}{3}[-1,1,-1]$$

$$= \left[\frac{68}{21}, \frac{29}{42}, \frac{103}{42}\right].$$

Problem 4 First, find a basis for $\mathbb{R}^3$ containing the vector [1,1,1]. Once we do that, we can use the Gram-Schmidt process. We know that $\{[1,0,0],[0,1,0],[0,0,1]\}$ is a basis for $\mathbb{R}^3$ thus $\{[1,1,1],[1,0,0],[0,1,0],[0,0,1]\}$ spans $\mathbb{R}^3$ (it's not a basis since the vectors aren't independent. Form the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\checkmark \quad \checkmark \quad \checkmark \quad \uparrow$ no pivot

Thus the set $\{[1,1,1],[1,0,0],[0,1,0]\}$ is a basis for $\mathbb{R}^3$. Finally, use the GramSchmidt command in Maple to obtain an orthogonal basis: $\{[1,1,1],[2/3,-1/3,-1/3],[0,1/2,-1/2]\}$.

$\boxed{\text{Problem 5}}$ Suppose that $A$ is an orthogonal matrix.

Then (as we have seen in class and in a homework problem), for any vectors x and y (of the right size so that the multiplication makes sense),

$$A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$$

$$\Rightarrow A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x}$$

$$\|A\vec{x}\|^2 = \|\vec{x}\|^2$$

$$\|A\vec{x}\| = \|\vec{x}\|.$$

$\boxed{\text{Problem 6.}}$ Suppose that $P$ is a projection matrix and that $\lambda$ is an eigenvalue of $P$ with corresponding eigenvector $\vec{v}$.

$$\Rightarrow P\vec{v} = \lambda\vec{v}$$

$$P^2\vec{v} = \lambda^2\vec{v}.$$

Since $P^2 = P$ for any projection matrix,

$$P\vec{v} = P^2\vec{v}, \text{ so } \lambda\vec{v} = \lambda^2\vec{v}$$

Thus we conclude that

$$\lambda^2 = \lambda,$$

so the only possible eigenvalues of $P$ are

$\lambda = 0$ and $\lambda = 1$.

As review for the exam, you should think geometrically about why this is true. In particular, suppose that $P$ is the projection matrix for a subspace $W$ of $\mathbb{R}^n$.

Prove the following:

(i) If $\vec{v}$ is in $W$, then $P\vec{v} = \vec{v}$, so $\vec{v}$ is an eigenvector of $P$ with eigenvalue 1.

(ii) If $\vec{v}$ is in $W^\perp$, then $P\vec{v} = \vec{0}$, so $\vec{v}$ is an eigenvector of $P$ with eigenvalue 0.

(iii) If $\vec{v}$ is not in $W$ or $W^\perp$, then $\vec{v}$ cannot be an eigenvector of $P$.

Thus the only possible eigenvalues of $P$ are $\lambda = 0$ and $\lambda = 1$, corresponding to cases (i) and (ii) above.

**Problem 7** The rank of $P$ is equal (by definition) to the dimension of the column space of $P$. The column space of $P$ is the set of all linear combinations of the columns of $P$. For every vector $\vec{v}$ in $\mathbb{R}^n$,

$$P\vec{v} \text{ is in } W.$$

Since $\dim(W) = 3$, the column space of $P$ has dimension 3. Thus:

$$\text{rank}(P) = 3.$$

Alternatively, think of $P$ as corresponding to a linear transformation and use the rank equation for linear transformations.

$\boxed{\text{Problem 8}}$ Suppose that $P$ is a projection matrix for a subspace $W$ of $\mathbb{R}^n$. Then for any vector $\vec{v}$ in $\mathbb{R}^n$, $P\vec{v} = \vec{v}_W$, the projection of $\vec{v}$ on the subspace $W$.

Then
$$\begin{aligned} P^2\vec{v} &= P(P\vec{v}) \\ &= P\vec{v}_W \\ &= \vec{v}_W, \end{aligned}$$

since $\vec{v}_W$ is already in $W$, i.e. the projection of $\vec{v}_W$ on $W$ is just $\vec{v}_W$. Thus, for every vector $\vec{v}$ in $\mathbb{R}^n$,
$$P^2\vec{v} = P\vec{v}$$
$$(P^2 - P)\vec{v} = \vec{0}.$$

Since this is true <u>for every</u> vector $\vec{v}$ in $\mathbb{R}^n$,

$P^2 - P = 0$, so $P^2 = P$.

$\boxed{\text{Problem 9.}}$ If $\{a_1, a_2, \ldots, a_k\}$ is an orthonormal basis for $\mathbb{R}^n$, then

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_k \\ | & | & & | \end{bmatrix}_{n \times k}$$ is an orthogonal matrix,

i.e. $A^T A = I$.

Then $P = A(A^T A)^{-1} A^T$

$= A \cdot I \cdot A^T$

$= AA^T$

Note that $A^T A = I$ does not imply $A^T = A^{-1}$ in this case since $A$ might not be a square matrix (and thus might not be invertible).

$\boxed{\text{Problem 10}}$ $\vec{v} = x + x^4$ in $P_4$.

$B = (1, 2x-1, x^3+x^4, 2x^3, x^2+2)$

consider the standard basis

$$B' = (x^4, x^3, x^2, x, 1)$$

$\vec{v}_{B'} = [1, 0, 0, 1, 0]$

$(1)_{B'} = [0, 0, 0, 0, 1]$ $(2x^3)_{B'} = [0, 2, 0, 0, 0]$

$(2x-1)_{B'} = [0, 0, 0, 2, 1]$ $(x^2+2)_{B'} = [0, 0, 1, 0, 2]$

$(x^3+x^4)_{B'} = [1, 1, 0, 0, 0]$

so we need to solve the linear system

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = r_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Thus we form the corresponding augmented matrix and row-reduce.

$$
\begin{array}{cccccc}
r_1 & r_2 & r_3 & r_4 & r_5 & \\
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 2 & 0
\end{array}\right]
$$

$$\downarrow$$

$$
\left[\begin{array}{c|c}
 & -1/2 \\
 & 1/2 \\
I_5 & 1 \\
 & -1/2 \\
 & 0
\end{array}\right]
$$

$\Rightarrow r_1 = -1/2 \qquad r_3 = 1 \qquad r_5 = 0$

$\quad r_2 = 1/2 \qquad r_4 = -1/2$

$\Rightarrow \vec{V}_B = [-1/2, 1/2, 1, -1/2, 0].$

$\boxed{\text{Problem 11}}$ Note: You won't need to know how to do this problem for Exam 3 (since we won't cover all of the details until class on Tuesday, 12/14), but you will need to be able to do it on the Final Exam.

First, note that $\dim(P_3) = 4$, and $B'$ contains 4 vectors, so we just need to determine if the vectors are independent. To do this, we coordinatize the vectors relative to the standard basis

$$B = (x^3, x^2, x, 1)$$ and determine whether the resulting vectors in $\mathbb{R}^4$ are independent.

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1 \Rightarrow ((x+1)^3)_B = [1, 3, 3, 1]$$

$$(x+1)^2 = x^2 + 2x + 1 \Rightarrow ((x+1)^2)_B = [0, 1, 2, 1]$$

$$(x+1)_B = [0, 0, 1, 0]$$

$$(1)_B = [0, 0, 0, 1].$$

To determine if these 4 vectors in $\mathbb{R}^4$ are independent, we form the matrix A whose columns are the vectors.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Since $\mathrm{rref}(A) = I_4$, the vectors are independent in $\mathbb{R}^4$.

Thus $B'$ is a basis for $P_3$

## Problem 12

(a). True: every vector space must contain a zero vector.

(b). False. The set $\{\vec{0}\}$ is a vector space.

(c). True. This number is equal to dim(V).

(d). False. (see below)

(e) True. The projection of $\vec{b}$ on $sp(\vec{a})$ is:

$$\frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\cdot\vec{a}.$$

(f). False. The zero vector is in $W$ and $W^\perp$.

(g). True. If $A^TA = B^TB = I$, then $(AB)^TAB =$
$B^TA^TAB = B^TIB = B^TB = I$.

(h) True. We have seen in a homework problem that every projection matrix $P$ satisfies $P^T = P$