Math 224
Linear Algebra

Practice Exam 2
Solutions.

1. $\det(A) = 7$

(a) $\det(3A) = 3 \times 3 \times 3 \times \det(A)$

$= 3^3 \times 7$ (since each row is multiplied by 3)

$= 189$

(b) $\det(A^{-1}) = (\det(A))^{-1}$

$= \frac{1}{7}$

(c) $\det(2A^{-1}) = 2 \times 2 \times 2 \times \det(A^{-1})$

$= \frac{8}{7}$

(d) $\det(2A) = 2 \times 2 \times 2 \times 7 = 56$, so $\det((2A)^{-1}) = \frac{1}{56}$

2. Suppose A is an invertible $n \times n$ matrix.

$\Rightarrow \det(A) \neq 0$

$\Rightarrow (\det(A))^2 \neq 0$.

$\Rightarrow \det(A) \cdot \det(A) \neq 0$

$\Rightarrow \det(A^T) \cdot \det(A) \neq 0$. since $\det(A^T) = \det(A)$

$\Rightarrow \det(A^T A) \neq 0$

$\Rightarrow A^T A$ is invertible.

Suppose $A^T A$ is invertible.

$\Rightarrow \det(A^T A) \neq 0$

$\Rightarrow \det(A^T)\det(A) \neq 0$

$\Rightarrow (\det(A))^2 \neq 0$

$\Rightarrow \det(A) \neq 0$

$\Rightarrow A$ is invertible.

3. $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & -1 & 4 & 1 \\ 0 & 3 & 2 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$

$V = |\det(A)| = |-12| = 12.$

4. $\vec{u} = [u_1, u_2, u_3]$ $\vec{v} = [v_1, v_2, v_3]$

$\vec{u} + k\vec{v} = [u_1 + kv_1, u_2 + kv_2, u_3 + kv_3]$

$(\vec{u} + k\vec{v}) \times \vec{v} = [v_3 u_2 - v_2 u_3, v_1 u_3 - v_3 u_1, v_2 u_1 - v_1 u_2]$

$= \vec{u} \times \vec{v}$

(by direct computation of the cross products).

5. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

The eigenvalues of A are $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 1$.

Eigenvectors:

$\underline{\lambda_1 = \lambda_2 = -2}$: $A - (-2)\underline{I} = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}$

algebraic multiplicity 2

rref $\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$V_2 = r \quad V_1 = -r \quad V_3 = 0$

$\Rightarrow \vec{V} = \begin{bmatrix} -r \\ r \\ 0 \end{bmatrix}$, $r \neq 0 \Rightarrow \dim(E_{-2}) = 1$.

A is not diagonalizable since the geometric multiplicity of $\lambda_1 = \lambda_2 = -2$ is 1 but the algebraic multiplicity is 2.

6. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

The eigenvalues of A are $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 1$.

Eigenvectors:

$\underline{\lambda_1 = \lambda_2 = -2}$, $A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$

rref $\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $v_2 = r$, $v_3 = s$

$v_1 = -r - s$.

$$\vec{V} = \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} \text{ ; } r, s \text{ not both } 0.$$

$$= r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ ; } r, s \text{ not both } 0.$$

$$\left.\begin{array}{l} r=1,\ s=0: \quad \vec{V}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ \\ r=0,\ s=1: \quad \vec{V}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array}\right\} \text{ 2 independent eigenvectors}$$

$\underline{\lambda_3 = 1}$: $A - 1 \cdot I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$

rref $\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$V_3 = r$

$V_1 = r$ $\quad \vec{V} = \begin{bmatrix} r \\ -r \\ r \end{bmatrix}$, $r \neq 0$.

$V_2 = -r$

$r = 1 \Rightarrow \vec{V_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$$C = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$C^{-1}AC = D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = CDC^{-1}$$

$$A^k = CD^kC^{-1}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} C^{-1}$$

$$\Rightarrow A^k = \begin{bmatrix} 1 & -(-2)^k+1 & -(-2)^k+1 \\ (-2)^k-1 & 2(-2)^k-1 & (-2)^k-1 \\ -(-2)^k+1 & -(-2)^k+1 & 1 \end{bmatrix}$$

7. $p(\lambda)$ is a degree 6 polynomial, so A is a 6×6 matrix. Since $\lambda = 0$ is not an eigenvalue of A, A is invertible.

8. Suppose that $\vec{x}$ is an eigenvector of AB with corresponding eigenvalue λ, and that $B\vec{x} \neq \vec{0}$.

$$\Rightarrow (AB)\vec{x} = \lambda \vec{x}.$$

Now consider $(BA)(B\vec{x})$.

$$(BA)(B\vec{x}) = B(AB)\vec{x}$$
$$= B \cdot \lambda \vec{x}$$
$$= \lambda \cdot B\vec{x},$$

so $B\vec{x}$ is an eigenvector of BA with corresponding eigenvalue λ.

9. Suppose A and B are similar matrices. Then there exists an invertible matrix C such that $C^{-1}AC = B$.

$$\det(A - \lambda I) = \det(C^{-1})\det(A - \lambda I)\det(C)$$
$$= \det(C^{-1}AC - C^{-1}\lambda I C)$$
$$= \det(B - \lambda I),$$

so A and B have the same characteristic polynomial, and thus the same eigenvalues.

10. (a) False. $\det(4A) = 4^n \det(A)$.

(b) False.

(c) True. If A is a square matrix,

$\det(A^T A) = (\det(A))^2$

(d) False.

(e) False.

(f) True.

(g) True.

(h) False.