Math 224
Homework 9 Solutions

Section 6.1. Note: The form of the answers for the problems in this section is not unique. I have written basis vectors using integers, when possible, but many of your answers will likely contain fractions. As long as you're finding orthogonal complements by finding the nullspace of the correct matrix, your answers will be equivalent.

6.1 #8: The orthogonal complement is the nullspace of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \end{bmatrix},$$

so it is

$$sp([1,2,0,0],[0,0,1,0],[3,0,0,2]).$$

6.1 #9: The orthogonal complement is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix},$$

so it is

$$sp([-12,4,5]).$$

6.1 #11: The orthogonal complement is the nullspace of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 0 & -2 & 1 \end{bmatrix},$$

so it is

$$sp([2,-7,1,0],[-1,-2,0,1]).$$

6.1 #15: Let $W = sp([3,1,2],[1,0,1]$. The orthogonal complement of W is the nullspace of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

Thus

$$W^{\perp} = sp([-1,1,1]).$$

So we need to write $[1,2,1]$ as a linear combination of $[3,1,2],[1,0,1]$ and $[-1,1,1]$. Thus we form the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{array}\right],$$

and row reduce to obtain

$$[1,2,1] = \frac{4}{3}[3,1,2] + \frac{-7}{3}[1,0,1] + \frac{2}{3}[-1,1,1].$$

Thus

$$\mathbf{b_W} = \frac{4}{3}[3,1,2] + \frac{-7}{3}[1,0,1] = [5/3, 4/3, 1/3].$$

6.1 #17: Let $W = sp([2,1,1],[1,0,2])$. The orthogonal complement of W is the nullspace of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Thus

$$W^{\perp} = sp([-2,3,1]).$$

So we need to write $[1,0,0]$ as a linear combination of $[2,1,1],[1,0,2]$ and $[-2,3,1]$. Thus we form the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right],$$

and row reduce to obtain

$$[1,0,0] = \frac{3}{7}[2,1,1] + \frac{-1}{7}[1,0,2] + \frac{-1}{7}[-2,3,1].$$

Thus

$$\mathbf{b_W} = \frac{3}{7}[2,1,1] + \frac{-1}{7}[1,0,2] = [5/7, 3/7, 1/7].$$

6.1 #20: Since $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ form an orthonormal basis for $\mathbb{R}^4$, the projection of any vector $\mathbf{v}$ on subspaces generated by one or more of these vectors is obtained by keeping the corresponding components of $\mathbf{v}$ and replacing the other components by zeros.

(a) $[0,0,3,0]$

(b) $[-2,0,0,-5]$

(c) $[-2,0,3,-5]$

(d) $[-2,1,3,-5]$

6.1 #25: Suppose that $\mathbf{p_1}$ and $\mathbf{p_2}$ are two vectors in W such that $\mathbf{b} - \mathbf{p_1}$ and $\mathbf{b} - \mathbf{p_2}$ are orthogonal to every vector in W. Now, since $\mathbf{p_1}$ and $\mathbf{p_2}$ are in W, $\mathbf{p_1} - \mathbf{p_2}$ is also in W. Thus:

$$\begin{aligned}(\mathbf{p_1}-\mathbf{p_2})\cdot\mathbf{w} &= [(\mathbf{b}-\mathbf{p_1})-(\mathbf{b}-\mathbf{p_2})]\cdot\mathbf{w}\\ &= (\mathbf{b}-\mathbf{p_2})\cdot\mathbf{w}-(\mathbf{b}-\mathbf{p_1})\cdot\mathbf{w}\\ &= 0-0\\ &= 0\end{aligned}$$

for all vectors $\mathbf{w}$ in W. Thus, $\mathbf{p_1}-\mathbf{p_2}$ is orthogonal to every vector in W. Since, in particular, $\mathbf{p_1}-\mathbf{p_2}$ is in W, we have

$$(\mathbf{p_1}-\mathbf{p_2})\cdot(\mathbf{p_1}-\mathbf{p_2})=0.$$

Thus

$$||\mathbf{p_1}-\mathbf{p_2}||=0,$$

so $\mathbf{p_1}-\mathbf{p_2}=\mathbf{0}$. So $\mathbf{p_1}=\mathbf{p_2}$, as needed.

Section 6.2

6.2 #4: First, check by computing dot products that the given set of vectors is orthogonal. For convenience of notation, let $\mathbf{v_1}=[1,-1,1,1],\mathbf{v_2}=[-1,1,1,1],\mathbf{v_3}=[-1,1,1,1]$. Then

$$\begin{aligned}\mathbf{b_W} &= \frac{\mathbf{bv_1}}{\mathbf{v_1v_1}}\mathbf{v_1}+\frac{\mathbf{bv_2}}{\mathbf{v_2v_2}}\mathbf{v_2}+\frac{\mathbf{bv_3}}{\mathbf{v_3v_3}}\mathbf{v_3}\\ &= 0[1,-1,1,1]+3/2[-1,1,1,1]+3/2[1,1,-1,1]\\ &= [0,3,0,3]\end{aligned}$$

6.2 #8: We apply the Gram-Schmidt process. Let $\mathbf{a_1}=[1,1,0],\mathbf{a_2}=[-1,2,1]$. First,

$$\mathbf{v_1}=\mathbf{a_1}=[1,1,0].$$

Next,

$$\begin{aligned}\mathbf{v_2} &= \mathbf{a_2}-\frac{\mathbf{a_2v_1}}{\mathbf{v_1v_1}}\mathbf{v_1}\\ &= [-1,2,1]-1/2[1,1,0]\\ &= [-3/2,3/2,1]\end{aligned}$$

Thus an orthogonal basis is

$$\{[1,1,0],[-3/2,3/2,1]\}.$$

Normalizing, an orthonormal basis is

$$\{[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0], [\frac{-3}{\sqrt{22}}, \frac{3}{\sqrt{22}}, \frac{2}{\sqrt{22}}]\}.$$

6.2 #19: First, we find a basis for the nullspace of

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 5 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}.$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so a basis for the nullspace of A is

$$\{[-3, 1, 1, 0], [3, -2, 0, 1]\}.$$

Next, we apply the Gram-Schmidt process to this basis. We have

$$\mathbf{v_1} = [-3, 1, 1, 0]$$

and

$$\begin{aligned} \mathbf{v_2} &= [3, -2, 0, 1] - \frac{[3, -2, 0, 1][-3, 1, 1, 0]}{[-3, 1, 1, 0][-3, 1, 1, 0]}[-3, 1, 1, 0] \\ &= [0, -1, 1, 1] \end{aligned}$$

So an orthogonal basis for the nullspace of A is

$$\{[-3, 1, 1, 0], [0, -1, 1, 1]\}.$$

6.2 #20: First, we augment $\mathbf{a} = [1, 1, 1]$ to a basis for $\mathbb{R}^3$. A basis for $\mathbf{R}^3$ containing $[1, 1, 1]$ is the set

$$\{[1, 1, 1], [1, 0, 0], [0, 1, 0], [0, 0, 1]\}.$$

As in Section 2.1, we form the matrix A whose columns are these vectors, and row-reduce:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Since the fourth column does not contain a pivot, the set

$$\{\mathbf{a}, [1,0,0], [0,1,0]\}$$

is a basis for $\mathbf{R}^3$. Next, we apply the Gram-Schmidt process to this set. Let

$$\mathbf{v_1} = [1,1,1].$$

Then

$$\mathbf{v_2} = [2/3, -1/3, -1/3]$$

and

$$\mathbf{v_3} = [0, 1/2, -1/2].$$

Finally, we normalize to find an orthonormal basis:

$$\{[\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}], [\frac{\sqrt{6}}{3}, \frac{-\sqrt{6}}{6}, \frac{-\sqrt{6}}{6}], [0, \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}]\}.$$