Math 224
Homework 8 Solutions

Section 5.2

5.2 #2 The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = (\lambda - 2)(\lambda - 5),$$

so the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 5$. The eigenvectors corresponding to λ_1 are $\mathbf{v_1} = \begin{bmatrix} -2r \\ r \end{bmatrix}$, $r \neq 0$. The eigenvectors corresponding to $\lambda_2 = 5$ are $\mathbf{v_2} = \begin{bmatrix} s \\ s \end{bmatrix}$, $s \neq 0$. Taking $r = s = 1$, we find that $C^{-1}AC = D$, where

$$C = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Note that other matrices C are possible by taking different values for r and s.

5.2 #6 The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = -(\lambda + 1)(\lambda - 2)(\lambda - 3),$$

so the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The eigenvectors corresponding to $\lambda_1 = -1$ are $\mathbf{v_1} = \begin{bmatrix} -5r \\ 2r \\ r \end{bmatrix}$, $r \neq 0$. The eigenvectors corresponding to $\lambda_2 = 2$ are $\mathbf{v_2} = \begin{bmatrix} -3s \\ s \\ s \end{bmatrix}$, $s \neq 0$. The eigenvectors corresponding to $\lambda_3 = 3$ are $\mathbf{v_3} = \begin{bmatrix} -5t \\ 2t \\ 2t \end{bmatrix}$, $t \neq 0$. Taking $r = s = t = 1$, we find that $C^{-1}AC = D$, where

$$C = \begin{bmatrix} -5 & -3 & -3 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Note that other matrices C are possible by taking different values for r, s, and t.

5.2 #10 The matrix is not diagonalizable since the eigenvalue 3 has algebraic multiplicity 3 but geometric multiplicity 1.

5.2 #12 The matrix is diagonalizable since it is symmetric.

5.2 #14 $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ and $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

5.2 #15 Assume that A is a square matrix such that $D = C^{-1}AC$ is a diagonal matrix for some invertible matrix C. Since $CC^{-1} = I$, we have $(CC^{-1})^T = (C^{-1})^T C^T = I^T = I$, so $(C^T)^{-1} = (C^{-1})^T$. Thus $D^T = (C^{-1}AC)^T = C^T A^T (C^T)^{-1}$, so A^T is similar to the diagonal matrix D^T.

5.2 #17 $p(\lambda) = \det(A - \lambda I)$, so $p(0) = \det(A)$. On the other hand, $p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$, so $p(0) = (-1)^{2n}\lambda_1\lambda_2\cdots\lambda_n$. Thus $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

Section 5.3

5.3 #10 $A = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$, and $p(\lambda) = -\lambda(\lambda+3)(\lambda-1)$. Thus the eigenvalues of A are $\lambda_1 = -3$, $\lambda_2 = 0$, $\lambda_3 = 1$. The corresponding eigenvectors are $\mathbf{v_1} = \begin{bmatrix} r \\ -r \\ 2r \end{bmatrix}$, $r \neq 0$; $\mathbf{v_2} = \begin{bmatrix} -s \\ 2s \\ 0 \end{bmatrix}$, $s \neq 0$; $\mathbf{v_3} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$, $t \neq 0$. Letting $r = s = t = 1$, we obtain $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$. Then $C^{-1}AC = D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and we obtain the modified system $\mathbf{y}' = D\mathbf{y}$. The solution to this system is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_1 e^{-3t} \\ K_2 \\ K_3 e^t \end{bmatrix}$. Thus the solution to the original system is $\mathbf{x} = C\mathbf{y}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_1 e^{-3t} + K_2 \\ -K_1 e^{-3t} + 2K_2 + K_3 e^{3t} \\ 2K_1 e^{-3t} + K_3 e^t \end{bmatrix}.$$

5.3 #12 $A = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$, and $p(\lambda) = (\lambda+1)(\lambda-2)(\lambda-3)$. Thus the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The corresponding eigenvectors are

$\mathbf{v_1} = \begin{bmatrix} -5r \\ 2r \\ r \end{bmatrix}$, $r \neq 0$; $\mathbf{v_2} = \begin{bmatrix} -3s \\ s \\ s \end{bmatrix}$, $s \neq 0$; $\mathbf{v_3} = \begin{bmatrix} -5t \\ 2t \\ 2t \end{bmatrix}$, $t \neq 0$. Letting $r = s = t = 1$, we obtain $C = \begin{bmatrix} -5 & -3 & -5 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. Then $C^{-1}AC = D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and we obtain the modified system $\mathbf{y}' = D\mathbf{y}$. The solution to this system is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_1e^{-t} \\ K_2e^{2t} \\ K_3e^{3t} \end{bmatrix}$. Thus the solution to the original system is $\mathbf{x} = C\mathbf{y}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5K_1e^{-t} - 3K_2e^{2t} - 5K_3e^{3t} \\ 2K_1e^{-t} + K_2e^{2t} + 2K_3e^{3t} \\ K_1e^{-t} + K_2e^{2t} + 2K_3e^{3t} \end{bmatrix}.$$