Math 224
Homework 7 Solutions

Section 5.1

5.1 #2 $p(\lambda) = \det(A - \lambda I) = (\lambda + 3)(\lambda - 2)$, so the eigenvalues of A are $\boxed{\lambda_1 = -3 \text{ and } \lambda_2 = 2}$. Next, we compute the eigenvectors of A.

$$(A - (-3)I) = \begin{bmatrix} 10 & 5 \\ 10 & -5 \end{bmatrix},$$

which is row equivalent to $\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$. Since the second column does not contain a pivot, v_2 is a free variable, so we set $v_2 = r$. Then the first row implies $v_1 = -\frac{1}{2}r$. So the eigenvector corresponding to $\lambda_1 = -3$ is $\boxed{\mathbf{v_1} = \begin{bmatrix} -\frac{1}{2}r \\ r \end{bmatrix}, r \neq 0}$.

For $\lambda_2 = 2$, we the row-reduced form of $(A - 2I)$ is

$$\begin{bmatrix} 5 & 5 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since the second column does not contain a pivot, v_2 is a free variable, so we set $v_2 = s$. Then the first row implies $v_1 = -s$. So the eigenvector corresponding to $\lambda_2 = 2$ is $\boxed{\mathbf{v_2} = \begin{bmatrix} -s \\ s \end{bmatrix}, s \neq 0}$.

5.1 #14 $p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 12\lambda^2 - 48\lambda + 64 = (4 - \lambda)^3$. The roots of $p(\lambda) = 0$ are $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = 4}$.

$$(A - 4I) = \begin{bmatrix} 0 & 0 & 0 \\ 8 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix},$$

which is row equivalent to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The second and third columns do not contain a pivot, so we set $v_2 = r$ and $v_3 = s$. Then the first row implies $v_1 = -s$. Thus the eigenvectors corresponding to $\lambda_1 = \lambda_2 = \lambda_3 = 4$ are $\boxed{\begin{bmatrix} -r \\ s \\ r \end{bmatrix}, r \text{ and } s \text{ not both } 0}$.

5.1 #22 The standard matrix representation of T is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 2 & -1 \\ 2 & 1 & 4 \end{bmatrix}.$$

$p(\lambda) = \det(A - \lambda I) = (3 - \lambda)(\lambda - 1)(\lambda - 5)$, so the eigenvalues of A (and hence, the eigenvalues of T are $\boxed{\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 5}$. The eigenvector corresponding to λ_1 is $\boxed{\mathbf{v_1} = \begin{bmatrix} -r \\ -r \\ r \end{bmatrix},\ r \neq 0}$. The eigenvector corresponding to λ_2 is $\boxed{\mathbf{v_2} = \begin{bmatrix} -s \\ s \\ s \end{bmatrix},\ s \neq 0}$. The eigenvector corresponding to λ_3 is $\boxed{\mathbf{v_3} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix},\ t \neq 0}$.

5.1 #27

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A^2\mathbf{v} &= AA\mathbf{v} \\ &= A\lambda\mathbf{v} \\ &= \lambda^2\mathbf{v} \end{aligned}$$

Continuing this process, we obtain

$$A^k\mathbf{v} = \lambda^k\mathbf{v},$$

so that λ^k is an eigenvalue of A^k with eigenvector $\mathbf{v}$ for any positive integer k.

5.1 #28 Let $A\mathbf{v} = \lambda\mathbf{v}$ and assume that A is invertible. Then $\lambda \neq 0$ since $\det(A) = \det(A - 0I) \neq 0$. Thus we have

$$\begin{aligned} \mathbf{v} &= A^{-1}(A\mathbf{v}) \\ &= A^{-1}(\lambda\mathbf{v}) \\ &= \lambda(A^{-1}\mathbf{v}) \end{aligned}$$

Thus

$$A^{-1}\mathbf{v} = (1/\lambda)\mathbf{v},$$

so $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{v}$.

5.1 #29 The zero vector is in E_λ, so E_λ is non-empty (remember that a subspace must be non-empty). Now suppose that $\mathbf{v}$ and $\mathbf{w}$ are in E_λ. Then

$$\begin{aligned} A(\mathbf{v}+\mathbf{w}) &= A\mathbf{v}+A\mathbf{w} \\ &= \lambda\mathbf{v}+\lambda\mathbf{w} \\ &= \lambda(\mathbf{v}+\mathbf{w}), \end{aligned}$$

so $\mathbf{v}+\mathbf{w}$ is in E_λ. For any scalar r, we have $A(r\mathbf{v}) = r(A\mathbf{v}) = r\lambda\mathbf{v} = \lambda(r\mathbf{v})$, so $r\mathbf{v}$ is in E_λ. Thus E_λ is a subspace of $\mathbf{R}^n$. Note that we could also just say that E_λ is the solution space of the homogeneous linear system $(A-\lambda I)\mathbf{v} = \mathbf{0}$, and is thus a subspace.

5.1 #30 We know that a square matrix A is invertible if and only if $\det(A) \neq 0$. Note that

$$\det(A) = \det(A - 0I),$$

so $\det(A) \neq 0$ if and only if 0 is not an eigenvalue of A.

5.1 #32 Suppose that λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then

$$\begin{aligned} (A+rI)\mathbf{v} &= A\mathbf{v}+rI\mathbf{v} \\ &= \lambda\mathbf{v}+r\mathbf{v} \\ &= (\lambda+r)\mathbf{v}. \end{aligned}$$

Thus $\lambda + r$ is an eigenvalue of $A + rI$ with corresponding eigenvector $\mathbf{v}$, so the eigenvalues of $A + rI$ are those of A increased by r, while the corresponding eigenvalues remain the same.

5.1 #38

$$\begin{aligned} p(\lambda) &= \det(A-\lambda I) \\ &= \det(A-\lambda I)(\det(C))^{-1}(\det(C)) \\ &= \det(A-\lambda I)\det(C^{-1})\det(C) \\ &= \det(C^{-1})\det(A-\lambda I)\det(C) \\ &= \det(C^{-1}AC-\lambda I) \end{aligned}$$

Thus A and $C^{-1}AC$ have the same characteristic polynomial, and hence the same eigenvalues.

5.1 #39 $p(\lambda) = \det(A - \lambda I) = \lambda^2 - 5\lambda + 7$, so we compute

$$A^2 - 5A + 7I = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + 7\begin{bmatrix} 1 & -0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

thus illustrating the Cayley-Hamilton theorem.