Math 224

Homework 5

Solutions.

Section 4.1 # 4, 8, 10, 12, 18, 22, 30, 38, 46, 50, 54.

$\boxed{4.1 \# 4}$ $21 \cdot 7 - (-4 \cdot 10) = \boxed{187}$

$\boxed{4.1 \# 8}$
$$\begin{vmatrix} 1 & -2 & 7 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} - -2 \cdot \begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} + 7 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1 \cdot 3 + 2 \cdot -4 + 7 \cdot -1 = 3 - 8 - 7 = \boxed{-12}$$

$\boxed{4.1 \# 10}$ (a)
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(a_2 c_3 - a_3 c_2) - a_2(a_1 c_3 - a_3 c_1) + a_3(a_1 c_2 - a_2 c_1) = 0.$$

(b)
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = a_1(b_2 a_3 - b_3 a_2) - a_2(b_1 a_3 - b_3 a_1) + a_3(b_1 a_2 - b_2 a_1) = 0.$$

4.1 #12

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= -[a_1(c_2b_3 - c_3b_2) - a_2(c_1b_3 - c_3b_1) + a_3(\cancel{b_2}\, c_1b_2 - c_2b_1)]$$

$$= - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

4.1 #18

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & -1 \\ 4 & -6 & 1 \end{vmatrix} = (3-6)\vec{i} - (-2+4)\vec{j} + (12-12)\vec{k}$$

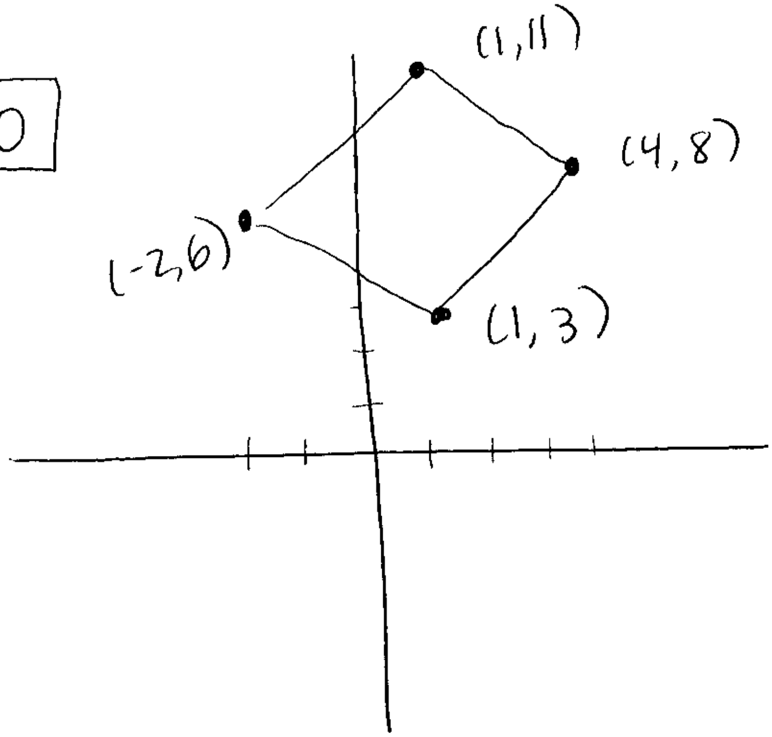
$$\boxed{= -3\vec{i} - 2\vec{j} = [-3, -2, 0]}$$

4.1 # 22 $A = \|\vec{a} \times \vec{b}\|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -5 \\ 2 & 4 & -1 \end{vmatrix} = 17\vec{i} - 9\vec{j} - 2\vec{k}$$

$$\|17\vec{i} - 9\vec{j} - 2\vec{k}\| = \sqrt{17^2 + (-9)^2 + (-2)^2} = \boxed{\sqrt{374}}$$

4.1 # 30



Notice that the area of the parallelogram is just the area of the parallelogram formed by the vectors

$\vec{a} = [-2, 6] - [1, 3] = [-3, 3]$ and

$\vec{b} = [4, 8] - [1, 3] = [3, 5]$.

So $A = \left| \begin{vmatrix} -3 & 3 \\ 3 & 5 \end{vmatrix} \right| = |-15 - 9| = |-24| = \boxed{24}$

$\boxed{4.1 \# 38}$

$$\begin{vmatrix} 2 & 1 & -4 \\ 3 & -1 & 2 \\ 1 & 3 & -8 \end{vmatrix} = 2(8-6) - 1(-24-2) + (-4)(9+1)$$

$$= -10$$

$\Rightarrow$ Volume $= |-10| = \boxed{10}$

$\boxed{4.1 \# 46}$ Three points are collinear if and only if the parallelogram obtained by the vectors from one point to the two others has area 0.

Let $\vec{a} = [4,2] - [0,0] = [4,2]$

$\vec{b} = [-6,-3] - [0,0] = [-6,-3]$

$$\begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} = -12 - -12 = 0,$$ so the points are collinear.

$\boxed{4.1\ \#50}$ Four points are coplanar if and only if the box obtained by the vectors from one point to the other three has volume 0.

let $\vec{a} = [2,1,1] - [0,0,0] = [2,1,1]$

$\vec{b} = [3,-2,1] - [0,0,0] = [3,-2,1]$

$\vec{c} = [-1,2,-3] - [0,0,0] = [-1,2,-3]$

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 2(-6-2) - 1(9+1) + 1(6-2)$$

$= -22 \neq 0$, so the ~~vectors~~ points are not coplanar.

$\boxed{4.1\ \#54}$ $(\vec{b} \times \vec{c}) - (\vec{c} \times \vec{b}) = (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{c})$

$= 2(\vec{b} \times \vec{c})$.

Section 4.2 # 8, 16, 18, 20, 24, 28, 32.

4.2 #8 Expand by minors down the 3rd column.

$$\begin{vmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 6 & 1 & 4 \\ 8 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 0 & 7 \\ 6 & 1 & 4 \\ 4 & 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 6 & 1 & 4 \\ 8 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix} \underset{\substack{\text{expand} \\ \text{down} \\ \text{col. 3}}}{=} 4 \cdot \begin{vmatrix} 8 & -2 \\ 4 & 1 \end{vmatrix} - 0 + 2 \cdot \begin{vmatrix} 6 & 1 \\ 8 & -2 \end{vmatrix}$$

$$= 4 \cdot (8+8) + 2 \cdot (-12-8)$$

$$= 4 \cdot 16 + 2 \cdot -20 = 24$$

$$\begin{vmatrix} 2 & 0 & 7 \\ 6 & 1 & 4 \\ 4 & 1 & 2 \end{vmatrix} \underset{\substack{\text{expand} \\ \text{down} \\ \text{col. 2}}}{=} -0 + 1 \cdot \begin{vmatrix} 2 & 7 \\ 4 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 7 \\ 6 & 4 \end{vmatrix}$$

$$= 1 \cdot (4-28) - 1 \cdot (8-42)$$

$$= -24 + 34 = 10$$

$$\Rightarrow \; = -1 \cdot 24 + 1 \cdot 10 = \boxed{-14}$$

$\boxed{4.2 \# 16}$ $\det(A^k) = \underbrace{\det(A) \cdot \det(A) \cdots \det(A)}_{k \text{ factors}}$

$= \boxed{2^k}$

$\boxed{4.2 \# 18}$ $\det(A+A) = \det(2A) = 2 \cdot 2 \cdot 2 \det(A) = 8 \cdot 2 = \boxed{16}$

since $2A$ means we multiply each of the three rows of A by 2 (thus multiplying $\det(A)$ by 2 three times).

$\boxed{4.2 \# 20}$ $\det(A^T) = \det(A) = \boxed{2}$

$\boxed{4.2 \# 24}$ $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} — & \vec{a} & — \\ — & \vec{b} & — \\ — & \vec{c} & — \end{bmatrix}$

$\det(A) = 3$

$$\cancel{M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 2a_1 + 3b_1 & & \end{bmatrix}}$$

Now consider the matrix

$$M = \begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & 2\vec{a}+3\vec{b}+2\vec{c} & - \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 2a_1+3b_1+2c_1 & 2a_2+3b_2+2c_2 & 2a_3+3b_3+2c_3 \end{bmatrix}$$

$$\det(M) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 2c_1 & 2c_2 & 2c_3 \\ \cancel{3b_1} & \cancel{3b_2} & \cancel{3b_3} \end{vmatrix}$$

(Add $-2 \times R1$ ~~to~~ and $3 \times R2$ to $R3$).

$$= 2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 2 \times 3 = \boxed{6}$$

$\boxed{4.2\#28}$ $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{vmatrix}$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} - 0 + 0$$

$$= (2-\lambda)[(1-\lambda)^2 - 4] = (2-\lambda)(\lambda^2 - 2\lambda - 3)$$

$$= (2-\lambda)(\lambda-3)(\lambda+1) = 0 \text{ for } \boxed{\lambda = 2, 3, -1.}$$

4.2 # 32: $\det(C^{-1}AC) = \det(C^{-1})\det(A)\det(C)$

$$= \det(C^{-1})\det(C)\det(A) = \det(C^{-1}C)\det(A)$$

$$= \det(I)\det(A) = \det(IA) = \det(A).$$

Alternatively, from the in-class worksheet,

$$\det(C^{-1}) = \frac{1}{\det(C)}, \text{ so}$$

$$\det(C^{-1})\det(A)\det(C) = \frac{1}{\det(C)}\det(C)\det(A) = \det(A).$$