Math 224
Homework 4 Solutions

Section 2.2

2.2 #4 $rref(A) = I_4$.

(a) rank(A)=4 since rref(A) has 4 columns with pivots.

(b) $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$

(c) $\{\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}\}$

(d) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, so $\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\}$ is a basis for the nullspace of A.

2.2 #6 $rref(A) = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) rank(A)=3

(b) $\{[1,0,0,-1/2],[0,1,0,1/2],[0,0,1,0]\}$

(c) $\{\begin{bmatrix} 0 \\ -4 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}\}$

(d) x_4 is a free variable since the 4th column does not contain a pivot. Set $x_4 = r$. Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$. So $\{\begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for the nullspace of A.

2.2 #8 $rref(A) = I_3$, so rank(A)=3=number of columns of A, so A is invertible.

2.2 #10 $rref(A) = I_4$, so rank(A)=4, so A is invertible.

Section 2.3

2.3 #4 T is not a linear transformation since $T([0,0]) \neq [0,0,0]$.

2.3 #8 Reducing $\left[\begin{array}{ccc|c} 1 & 0 & 0-1 & \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$, we find that

$$[-1,4,2] = -1 \cdot [1,0,0] + 6 \cdot [0,1,0] + 2 \cdot [0,-1,1],$$

so $T([-1,4,2]) = -1 \cdot T([1,0,0]) + 6 \cdot T([0,1,0)] + 2 \cdot T([0,-1,1]) = -1 \cdot [1,3] + 6 \cdot [4,-1] + 2 \cdot [3,-5] = \boxed{[31,-19]}$.

2.3 #10 $\left[\begin{array}{cc|c} -1 & 1 & x \\ 1 & 1 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & (-x+y)/2 \\ 0 & 1 & (x+y)/2 \end{array}\right]$. Thus

$$[x,y] = \frac{-x+y}{2}[-1,1] + \frac{x+y}{2}[1,1].$$

So

$$\begin{aligned} T([x,y]) &= \tfrac{-x+y}{2} T([-1,1]) + \tfrac{x+y}{2} T([1,1]) \\ &= \tfrac{-x+y}{2}[2,1,4] + \tfrac{x+y}{2}[-6,3,2] \\ &= \boxed{[-4x-2y, x+2y, -x+3y]}. \end{aligned}$$

2.3 #16 $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

2.3 #19 The matrix A associated with T is $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$. The matrix A' associated with T' is $A' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Thus the matrix associated with $T' \circ T$ is $A'A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$. So $(T' \circ T)([x_1, x_2]) = [2x_1, 3x_1 + x_2]$.

2.3 #20 Referring to the notation in #19, the matrix associated with $T \circ T'$ is $AA' = \begin{bmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$. So $(T \circ T')([x_1, x_2, x_3]) = [3x_1 - x_2 + 2x_3, x_1 - x_2 + x_3, -2x_2 + x_3]$.

2.3 #24 No, because A is not a square matrix, so it cannot be invertible.

2.3 #30 $T^{-1}(T([x_1, x_2, x_3])) =$
$[(x_1 - 2x_2 + x_3) + 4(x_2 - x_3) - 2(x_2 - 3x_3), 3(x_2 - x_3) - 2(x_2 - 3x_3), 2(x_2 - x_3) - (2x_2 - 3x_3)]$
$= [x_1, x_2, x_3]$.