Math 224
Homework 2 Solutions

Section 1.4

1.4 #6(a): $\begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 2 & 4 & -1 & 3 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 3 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$

1.4 #6(b): $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$

1.4#14: First we form the augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{cc|c} 4 & -3 & 10 \\ 8 & -1 & 10 \end{array}\right]$. Row reducing, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{cc|c} 4 & -3 & 10 \\ 0 & 5 & -10 \end{array}\right]$. The bottom row implies $5x_2 = -10$, so we conclude $\boxed{x_2 = -2}$. Back-substituting into the first row, we obtain $4x_1 - 3 \cdot -2 = 10$, so we conclude $\boxed{x_1 = 1}$.

1.4#18: First we form the augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 3 & -8 & 2 & 5 \end{array}\right]$. Row reducing, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right]$. The third column does not contain a pivot, so x_3 is a free variable, and we can assign any value, say s, to x_3. Next we back substitute to obtain x_1 and x_2 in terms of s. The second row implies $x_2 - x_3 = -1$, so $x_2 = -1 + s$. The first row implies $x_1 - 3x_2 + x_3 = 2$, so $x_1 = 2 + 3(-1 + s) - s = -1 + 2s$. So we conclude: $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 2s \\ -1 + s \\ s \end{bmatrix}}$.

1.4#22: The augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{cc|c} 2 & 8 & 16 \\ 5 & -4 & -8 \end{array}\right]$. Row reducing in Maple to reduced row-echelon form, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$. So we conclude $\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}}$.

1.4#24: The augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ 3 & 6 & -8 & -2 & 1 \end{array}\right]$. Row reducing in Maple to reduced row-echelon form, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -14 & \\ 0 & 0 & 1 & -5 & -5 \end{array}\right]$. The second and fourth columns do not contain pivots, so x_2 and x_4 are free variables. Set $x_2 = r$ and $x_4 = s$. Then we obtain: $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -13 - 2r + 14s \\ r \\ -5 + 5s \\ s \end{bmatrix}}$.

1.4#28: First we form the augmented matrix consisting of the column vectors v_i: $[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & -3 & 1 & 2 & 2 \\ 1 & -2 & 2 & 4 & -1 \\ 2 & -8 & -1 & 0 & 3 \\ 3 & -9 & 4 & 0 & 7 \end{array}\right]$. Row reducing in Maple to reduced row-echelon form, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -43 \\ 0 & 1 & 0 & 0 & -12 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$. Thus the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, so the vector $\mathbf{b}$ is in the span of the column vectors of A, so $\boxed{\mathbf{b} \text{ is in the span of the vectors } v_i}$.

1.4#34: The augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{cc|c} 1 & -5 & 13 \\ 3 & 2 & 5 \end{array}\right]$. Row reducing in Maple to reduced row-echelon form, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$. Thus we conclude $\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}}$.

1.4#42: Since the matrix on the right is obtained by replacing $R3$ with $-3 \cdot R1 + R3$, $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

1.4#44: $C = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

1.4#46: $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

1.4#56: We need to solve the system of equations:

$$\begin{aligned} a+b+c &= -4 \\ a-b+c &= 0 \\ 4a+2b+c &= 3 \end{aligned}$$

So we form the augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & 1 & 3 \end{array}\right]$. Row reducing in Maple to reduced row-echelon form, we obtain $[A|\mathbf{b}] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$. Thus we conclude $\boxed{a = 3,\ b = -2, \text{ and } c = 5}$, so the parabola is $\boxed{y = 3x^2 - 2x - 5}$.

Section 1.5

1.5#8(a): $A^{-1} = \begin{bmatrix} -36 & -24 & 13 \\ -19 & -13 & 7 \\ 3 & 2 & -1 \end{bmatrix}$

1.5#14: $A^{-1} = \begin{bmatrix} =7 & 5 & 3 \\ 3 & -2 & 2 \\ 3 & -2 & 1 \end{bmatrix}$. $A^{-1} \cdot \mathbf{b} = \begin{bmatrix} =7 & 5 & 3 \\ 3 & -2 & 2 \\ 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$. Thus we conclude $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}}$.

1.5#18: Since $2A = 2(IA) = A(2I)$, we see that we can take $B = 2I$, where I is the 3×3 identity matrix.

1.5#26: Since A^2 is invertible, we can let B be a matrix such that $A^2B = BA^2 = I$. Then $A(AB) = (BA)A = I$, so A is invertible and $A^{-1} = AB = BA = I$.

1.5#30(a): One example of such a matrix is $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

1.5#30(b): Suppose that A is a matrix that is both idempotent and invertible. Let A^{-1} denote the inverse of A. Then since A is idempotent, $A^2 = A$. Since A is invertible, we can multiply on the left by A^{-1} to obtain $A^{-1}AA = A^{-1}A$, which implies $IA = I$, so $A = I$.

1.5#35(a): $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{ad-bc} & 0 \\ 0 & \frac{ad-bc}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Similarly, $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So if $ad-bc \neq 0$, $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$ is the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.