Math 224
Homework 1 Solutions

Section 1.1

1.1 #8: $4(3\mathbf{u}+2\mathbf{v}-5\mathbf{w}) = 4([-3,\ 9,\ -6]+[8,\ 0,\ -2]-[15,\ -5,\ 10]) = \boxed{[80,\ 56,\ \text{-}72]}$

1.1 #26: $\boxed{\text{All } c \in \mathbb{R}}$. Since $[-3,5]$ and $[6,11]$ are not parallel, every vector $[-1,c]$ forms the diagonal of a parallelogram with adjacent sides formed by vectors parallel to $[-3,5]$ and to $[6,-11]$.

1.1 #28: If $[1,c,c-1] = r[1,2,1]+s[3,6,3]$, then $1 = r+3s$, $c = 2r+6s = 2(r+3s)$, and $c-1 = r+3s$ (by equating the components of the vectors on the left and right of the equation). The first and second equations yield $c = 2$, which does satisfy the third equation since $r+3s = 1$. Thus $\boxed{c=2}$.

1.1 #36: $x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$

Section 1.2

1.2 #4: $||\mathbf{v}-2\mathbf{u}|| = ||[2,1,-1]-[-2,6,8]|| = ||[4,-5,-9]||$
$= \sqrt{(4)^2+(-5)^2+(-9)^2} = \boxed{\sqrt{122}}$.

1.2 #8: $||\mathbf{w}|| = \sqrt{14}$.
$\frac{-1}{||\mathbf{w}||}\mathbf{w} = \frac{-1}{\sqrt{14}}[-2,1,3] = \boxed{\left[\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right]}$.

1.2 #30: $[4,1,2,1,6]\cdot[8,2,4,2,3] > 0$, so the vectors are $\boxed{\text{not perpendicular}}$. To check if the vectors are parallel, we must check if one is a constant multiple of the other. If $r[4,1,2,1,6] = [8,2,4,2,3]$, then $4r = 8$, so $r = 2$ (equating the first components). But equating the last components yields $6r = 3$, so $r = 1/2$. Since $2 \neq 1/2$, the vectors are $\boxed{\text{not parallel}}$.

1.2 #44: We know that $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, so $\mathbf{w}\cdot\mathbf{u} = 0$ and $\mathbf{w}\cdot\mathbf{v} = 0$. Thus:

$$\begin{aligned} \mathbf{w}\cdot(r\mathbf{u}+s\mathbf{v}) &= \mathbf{w}\cdot(r\mathbf{u}) + \mathbf{w}\cdot(s\mathbf{v}) \\ &= r(\mathbf{w}\cdot\mathbf{u}) + s(\mathbf{w}\cdot\mathbf{v}) \\ &= r(0)+s(0) \\ &= 0. \end{aligned}$$

Thus $\mathbf{w}$ is perpendicular to $r\mathbf{u}+s\mathbf{v}$.

Section 1.3

1.3 #8: Not possible since C is a 3×2 matrix and D is a 3×2 matrix.

1.3 #12: $AC = \begin{bmatrix} -13 & 14 \\ 11 & -6 \end{bmatrix}$ and $\boxed{(AC)^2 = \begin{bmatrix} 323 & -266 \\ -209 & 190 \end{bmatrix}}$.

1.3 #20: $\begin{bmatrix} 1 & -1 & 2 & 5 \\ -1 & 4 & -7 & 8 \\ 2 & -7 & -1 & 6 \\ 5 & 8 & 6 & 3 \end{bmatrix}$

1.3 #30: If $A = [a_{ij}]$ is an $m \times n$ matrix, then A^T is the $n \times m$ matrix whose (i, j)-th entry is a_{ji} and $(A^T)^T$ is the $m \times n$ matrix whose (i, j) entry is again a_{ij}. Thus $(A^T)^T = A$.

1.3 #32: The (i, j)-th entry of $(AB)^T$ is the (j, i)-th entry in AB, which is

$$\begin{aligned} & (j\text{-th row of } A) \cdot (i\text{th column of } B) \\ = & (i\text{-th column of } B) \cdot (j\text{-th row of } A) \\ = & (i\text{-th row of } A^T) \cdot (j\text{-th column of } A^T), \end{aligned}$$

which is the (i, j)-th entry of $B^T A^T$. Since $(AB)^T$ and $B^T A^T$ have the same size and the same entries, they are equal.

1.3 #38: Since $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$, we see that $A + A^T$ is symmetric.

1.3 #40(a): $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$ and $(AA^3)^T = (AA^2)^T = (A^2)^T A^T = (A^T)^2 A^T = (A^T)^3$.

1.3 #42: $(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2$, so $(A+B)^2 = A^2+2AB+B^2$ if and only if $AB = BA$. Since we know that matrix multiplication is, in general, not commutative, we conclude that $(A + B)^2 \neq A^2 + 2AB + B^2$ in general. In particular, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $(A + B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 + 2AB + B^2 = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$. The equation $(A + B)^2 = A^2 + 2AB + B^2$ for square matrices A and B is true if and only if A and B commute.

1.3 #46:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & r & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & r & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

These products are equal for $\boxed{\text{all values of } r}$.

Section 1.4

1.4 #2(a): $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 7 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 7 \end{bmatrix}$

Note: answers are not unique.

1.4 #4(a): $\begin{bmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 1 & 3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

Note: answers are not unique.

1.4 #14: First, we form the augmented matrix $\left[\begin{array}{cc|c} 4 & -3 & 10 \\ 8 & -1 & 10 \end{array}\right]$. Row reducing, we obtain the row-reduced matrix $\left[\begin{array}{cc|c} 4 & -3 & 10 \\ 0 & 5 & 10 \end{array}\right]$. Thus we obtain $5x_2 = -10$ and $4x_1 - 3x_2 = 10$. Thus the solution of the linear system is $\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$.