Math 224
Homework 10 Solutions

Section 6.3

6.3 #2: Let

$$A = \begin{bmatrix} 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$A^T A = I,$$

so A is orthogonal and

$$A^{-1} = A^T = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & 3/5 & 0 \end{bmatrix}.$$

6.3 #20: Suppose that A is orthogonal. Then A^T is also orthogonal, so $A^{-1} = A^T$ is also orthogonal. Thus

$$||A\mathbf{x}|| = ||\mathbf{x}|| \text{ and } ||A^{-1}\mathbf{x}|| = ||\mathbf{x}||.$$

Thus

$$||A\mathbf{x}|| = ||A^{-1}\mathbf{x}||$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.

6.3 #21: If A is orthogonal, then $A^T A = I$, so:

$$\begin{aligned} (A^2)^T A^2 &= I \\ &= (AA)^T AA \\ &= A^T A^T AA \\ &= A^T I A \\ &= A^T A \\ &= I \end{aligned}$$

Thus A^2 is also orthogonal.

6.3 #22: If A is orthogonal, then $A^T A = I$. Thus:

$$\begin{aligned} \det(A^T A) &= \det(I) \\ \det(A^T)\det(A) &= 1 \\ \det(A)\det(A) &= 1 \\ \det(A)^2 &= 1 \\ \det(A) &= \pm 1 \end{aligned}$$

6.3 #23: An example is the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

which has determinant 1 but is not orthogonal since

$$A^T A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \neq I.$$

6.3 #24: Since C is orthogonal, $C^{-1} = C^T$. We can rewrite $C^{-1}AC = D$ as $A = CDC^{-1}$. Thus:

$$\begin{aligned} A^T &= (CDC^{-1})^T \\ &= (CDC^T)^T \\ &= (C^T)^T D^T C^T \\ &= CDC^T \\ &= CDC^{-1} \\ &= A \end{aligned}$$

Thus A is symmetric.

6.3 #31: Since A and C are orthogonal, $A^T A = I$ and $C^T C = I$, so $A^{-1} = A^T$ and $C^{-1} = C^T$. Thus:

$$\begin{aligned} (C^{-1}AC)^T(C^{-1}AC) &= C^T A^T (C^{-1})^T C^{-1} AC \\ &= C^T A^T (C^T)^T C^T AC \\ &= C^T A^T C C^T AC \\ &= C^T A^T AC \\ &= C^T C \\ &= I \end{aligned}$$

Thus $C^{-1}AC$ is orthogonal.

Section 6.4

6.4#2: $A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, so

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1/6 & -1/6 & 1/3 \\ -1/6 & 1/6 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}.$$

The projection is

$$P\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

6.4 #8: $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$, so

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 3/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 3/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 3/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & 3/4 \end{bmatrix}.$$

The projection is

$$P\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 5/4 \\ 7/4 \\ 3/4 \end{bmatrix}.$$

6.4 #16: (a)

$$\begin{aligned} P^2 &= (A(A^TA)^{-1}A^T)^2 \\ &= (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) \\ &= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= P \end{aligned}$$

(b)

$$\begin{aligned} P^T &= (A(A^TA)^{-1}A^T)^T \\ &= (A^T)^T((A^TA)^{-1})^TA^T \\ &= A((A^TA)^T)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= P \end{aligned}$$

6.3 #21: If P is an invertible projection matrix, then $P^2 = P$ implies $P = I$ (since we can multiply both sides by P^{-1}). Thus the only invertible projection matrix is the identity matrix I.