Math 224
Some Important Information about Diagonalization

Let A be an $n \times n$ matrix, and let λ_1, λ_2, ..., λ_n be (possibly complex) scalars and let $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ be non-zero vectors in n-space (n-space means either $\mathbf{R}^n$ or $\mathbf{C}^n$). Let C be the $n \times n$ matrix having $\mathbf{v_j}$ as j-th column vector, and let D be the $n \times n$ matrix having the λ_i on the main diagonal and 0's elsewhere.

An $n \times n$ matrix A is **diagonalizable** if there exists an *invertible* matrix C such that $C^{-1}AC = D$, where D is some *diagonal* matrix. We say that the matrix C diagonalizes the matrix A.

Our goal: given a matrix A, decide whether or not A is diagonalizable, and if so, find matrices C and D such that

$$C^{-1}AC = D.$$

Note that if we can find such matrices C and D, then $A = CDC^{-1}$ and

$$A^k = CD^kC^{-1},$$

which is a useful result since computing powers of a diagonal matrix is easy.

1. $AC = CD$ if and only if λ_1, λ_2, ..., λ_n are eigenvalues of A with corresponding eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$.
2. Note that **if C is invertible**, then we can rewrite $AC = CD$ as $C^{-1}AC = D$, which was our original goal!! So, now our question is, when will C be invertible?
3. So, let's suppose that A has n eigenvectors and eigenvalues, and construct the matrices C and D using the eigenvalues λ_1, λ_2, ..., λ_n and eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$.
4. From the previous two statements, we know that A is diagonalizable if the rank of C is equal to n (i.e. C is invertible).
5. Equivalently, A is diagonalizable if the n eigenvectors of A that we have used to create C are independent.
6. So now, our question becomes: given a matrix A, when can we find n **independent eigenvectors** of A?
7. If $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ are eigenvectors of A corresponding to **distinct** eigenvalues λ_1, λ_2, ..., λ_n, then the eigenvectors are independent, so A is diagonalizable.

8. Note that the previous statement does NOT imply that if A does not have distinct eigenvalues, then we can't find n independent eigenvectors. The theorem just says that if the eigenvalues are distinct, then we are guaranteed that the corresponding eigenvectors are independent.

9. (Complete during/after class today). State the algebraic/geometric multiplicity criterion for A to be diagonalizable.

10. (Complete during/after class today). State the result about diagonalization of real symmetric matrices.