Tests for Convergence of Series

1) Use the comparison test to confirm the statements in the following exercises.

1. $\sum_{n=4}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=4}^{\infty} \frac{1}{n-3}$ diverges.

   Answer: Let $a_n = 1/(n-3)$, for $n \geq 4$. Since $n - 3 < n$, we have $1/(n-3) > 1/n$, so
   $$a_n > \frac{1}{n}.$$
   The harmonic series $\sum_{n=4}^{\infty} \frac{1}{n}$ diverges, so the comparison test tells us that the series $\sum_{n=4}^{\infty} \frac{1}{n-3}$ also diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ converges.

   Answer: Let $a_n = 1/(n^2+2)$. Since $n^2 + 2 > n^2$, we have $1/(n^2+2) < 1/n^2$, so
   $$0 < a_n < \frac{1}{n^2}.$$
   The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ also converges.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ converges.

   Answer: Let $a_n = e^{-n}/n^2$. Since $e^{-n} < 1$, for $n \geq 1$,we have $\frac{e^{-n}}{n^2} < \frac{1}{n^2}$, so
   $$0 < a_n < \frac{1}{n^2}.$$
   The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ also converges.

2) Use the comparison test to determine whether the series in the following exercises converge.

1. $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$

   Answer: Let $a_n = 1/(3^n+1)$. Since $3^n + 1 > 3^n$, we have $1/(3^n+1) < 1/3^n = \left(\frac{1}{3}\right)^n$, so
   $$0 < a_n < \left(\frac{1}{3}\right)^n.$$
   Thus we can compare the series $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$. This geometric series converges since $|1/3| < 1$, so the comparison test tells us that $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ also converges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$

   Answer: Let $a_n = 1/(n^4+e^n)$. Since $n^4 + e^n > n^4$, we have
   $$\frac{1}{n^4+e^n} < \frac{1}{n^4},$$
   so
   $$0 < a_n < \frac{1}{n^4}.$$
   Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$ also converges.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

   Answer: Since $\ln n \leq n$ for $n \geq 2$, we have $1/\ln n \geq 1/n$, so the series diverges by comparison with the harmonic series, $\sum 1/n$.

4. $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$

   Answer: Let $a_n = n^2/(n^4+1)$. Since $n^4+1 > n^4$, we have $\frac{1}{n^4+1} < \frac{1}{n^4}$, so

$$a_n = \frac{n^2}{n^4+1} < \frac{n^2}{n^4} = \frac{1}{n^2},$$

   therefore

$$0 < a_n < \frac{1}{n^2}.$$

   Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$ converges also.

5. $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3+1}$

   Answer: We know that $|\sin n| < 1$, so

$$\frac{n \sin^2 n}{n^3+1} \leq \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}.$$

   Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, comparison gives that $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3+1}$ converges.

6. $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$

   Answer: Let $a_n = (2^n+1)/(n2^n-1)$. Since $n2^n - 1 < n2^n + n = n(2^n+1)$, we have

$$\frac{2^n+1}{n2^n-1} > \frac{2^n+1}{n(2^n+1)} = \frac{1}{n}.$$

   Therefore, we can compare the series $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The comparison test tells us that $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ also diverges.

3) Use the ratio test to decide if the series in the following exercises converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

   Answer: Since $a_n = 1/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 1/(2n+2)!$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)},$$

   so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

   Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges.

2. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

   Answer: Since $a_n = (n!)^2/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = ((n+1)!)^2/(2n+2)!$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}.$$

   However, since $(n+1)! = (n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2(n!)^2(2n)!}{(2n+2)(2n+1)(2n)!(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{4n+2},$$

   so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}.$$

   Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

3. $\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$

Answer: Since $a_n = (2n)!/(n!(n+1)!)$, replacing n by $n+1$ gives $a_{n+1} = (2n+2)!/((n+1)!(n+2)!)$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(2n+2)!}{(n+1)!(n+2)!}}{\frac{(2n)!}{n!(n+1)!}} = \frac{(2n+2)!}{(n+1)!(n+2)!} \cdot \frac{n!(n+1)!}{(2n)!}.$$

However, since $(n+2)! = (n+2)(n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \frac{2(2n+1)}{n+2},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 4.$$

Since $L > 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$ diverges.

4. $\sum_{n=1}^{\infty} \frac{1}{r^n n!}, r > 0$

Answer: Since $a_n = 1/(r^n n!)$, replacing n by $n+1$ gives $a_{n+1} = 1/(r^{n+1}(n+1)!)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{r^{n+1}(n+1)!}}{\frac{1}{r^n n!}} = \frac{r^n n!}{r^{n+1}(n+1)!} = \frac{1}{r(n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{r} \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{r^n n!}$ converges for all $r > 0$.

5. $\sum_{n=1}^{\infty} \frac{1}{ne^n}$

Answer: Since $a_n = 1/(ne^n)$, replacing n by $n+1$ gives $a_{n+1} = 1/(n+1)e^{n+1}$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)e^{n+1}}}{\frac{1}{ne^n}} = \frac{ne^n}{(n+1)e^{n+1}} = \left(\frac{n}{n+1}\right)\frac{1}{e}.$$

Therefore

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{e} < 1.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{ne^n}$ converges.

6. $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$

Answer: Since $a_n = 2^n/(n^3+1)$, replacing n by $n+1$ gives $a_{n+1} = 2^{n+1}/((n+1)^3+1)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{(n+1)^3+1}}{\frac{2^n}{n^3+1}} = \frac{2^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{2^n} = 2\frac{n^3+1}{(n+1)^3+1},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2.$$

Since $L > 1$ the ratio test tells us that the series $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$ diverges.

4) Use the integral test to decide whether the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Answer: We use the integral test with $f(x) = 1/x^3$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^3}dx$ converges or diverges:

$$\int_1^{\infty} \frac{1}{x^3}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3}dx = \lim_{b\to\infty} \frac{-1}{2x^2}\bigg|_1^b = \lim_{b\to\infty} \left(\frac{-1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\displaystyle\int_1^{\infty} \frac{1}{x^3}dx$ converges, we conclude from the integral test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Answer: We use the integral test with $f(x) = x/(x^2+1)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\displaystyle\int_1^{\infty} \frac{x}{x^2+1} dx$ converges or diverges:

$$\int_1^{\infty} \frac{x}{x^2+1} dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1} dx = \lim_{b\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2\right) = \infty.$$

Since the integral $\displaystyle\int_1^{\infty} \frac{x}{x^2+1} dx$ diverges, we conclude from the integral test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

Answer : We use the integral test with $f(x) = 1/e^x$ to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\displaystyle\int_1^{\infty} \frac{1}{e^x} dx$ converges or diverges:

$$\int_1^{\infty} \frac{1}{e^x} dx = \lim_{b\to\infty} \int_1^b e^{-x} dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1}.$$

Since the integral $\displaystyle\int_1^{\infty} \frac{1}{e^x} dx$ converges, we conclude from the integral test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges. We can also observe that this is a geometric series with ratio $x = 1/e < 1$, and hence it converges.

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Answer: We use the integral test with $f(x) = 1/(x(\ln x)^2)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ converges or diverges:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \frac{-1}{\ln x}\Big|_2^b = \lim_{b\to\infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the integral $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ converges, we conclude from the integral test that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

5) Use the alternating series test to show that the following series converge.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Answer: Let $a_n = 1/\sqrt{n}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{n+1}$. Since $\sqrt{n+1} > \sqrt{n}$, we have $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

Answer: Let $a_n = 1/(2n+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(2n+3)$. Since $2n+3 > 2n+1$, we have

$$0 < a_{n+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ converges.

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$

Answer: Let $a_n = 1/(n^2+2n+1) = 1/(n+1)^2$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(n+2)^2$. Since $n+2 > n+1$, we have

$$\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2}$$

so

$$0 < a_{n+1} < a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$ converges.

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}$
   Answer: Let $a_n = 1/e^n$. Then replacing n by $n+1$ we have $a_{n+1} = 1/e^{n+1}$. Since $e^{n+1} > e^n$, we have $\frac{1}{e^{n+1}} < \frac{1}{e^n}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ converges by the alternating series test. We can also observe that the series is geometric with ratio $x = -1/e$ can hence converges since $|x| < 1$.

6) In the following exercises determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum \frac{(-1)^n}{2^n}$
   Answer: Both $\sum \frac{(-1)^n}{2^n} = \sum \left(\frac{-1}{2}\right)^n$ and $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ are convergent geometric series. Thus $\sum \frac{(-1)^n}{2^n}$ is absolutely convergent.

2. $\sum \frac{(-1)^n}{2n}$
   Answer: The series $\sum \frac{(-1)^n}{2n}$ converges by the alternating series test. However $\sum \frac{1}{2n}$ diverges because it is a multiple of the harmonic series. Thus $\sum \frac{(-1)^n}{2n}$ is conditionally convergent.

3. $\sum (-1)^n \left(1 + \frac{1}{n^2}\right)$
   Answer: Since
   $$\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right) = 1,$$
   the n^{th} term $a_n = (-1)^n \left(1 + \frac{1}{n^2}\right)$ does not tend to zero as $n \to \infty$. Thus, the series $\sum (-1)^n \left(1 + \frac{1}{n^2}\right)$ is divergent.

4. $\sum \frac{(-1)^n}{n^4+7}$
   Answer: The series $\sum \frac{(-1)^n}{n^4+7}$ converges by the alternating series test. Moreover, the series $\sum \frac{1}{n^4+7}$ converges by comparison with the convergent p-series $\sum \frac{1}{n^4}$. Thus $\sum \frac{(-1)^n}{n^4+7}$ is absolutely convergent.

5. $\sum \frac{(-1)^{n-1}}{n \ln n}$
   Answer: We first check absolute convergence by deciding whether $\sum 1/(n \ln n)$ converges by using the integral test. Since
   $$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b\to\infty} \ln(\ln(x))\Big|_2^b = \lim_{b\to\infty} (\ln(\ln(b)) - \ln(\ln(2))),$$
   and since this limit does not exist, $\sum \frac{1}{n \ln n}$ diverges.
   We now check conditional convergence. The original series is alternating so we check whether $a_{n+1} < a_n$. Consider $a_n = f(n)$, where $f(x) = 1/(x \ln x)$. Since
   $$\frac{d}{dx}\left(\frac{1}{x \ln x}\right) = \frac{-1}{x^2 \ln x}\left(1 + \frac{1}{\ln x}\right)$$
   is negative for $x > 1$, we know that a_n is decreasing for $n \geq 2$. Thus, for $n \geq 2$
   $$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n} = a_n.$$
   Since $1/(n \ln n) \to 0$ as $n \to \infty$, we see that $\sum \frac{(-1)^{n-1}}{n \ln n}$ is conditionally convergent.

6. $\sum \frac{(-1)^{n-1} \arctan(1/n)}{n^2}$
   Answer: We first check absolute convergence by deciding whether $\sum \frac{\arctan(1/n)}{n^2}$ converges. Since $\arctan x$ is the angle between $-\pi/2$ and $\pi/2$, we have $\arctan(1/n) < \pi/2$ for all n. We compare
   $$\frac{\arctan(1/n)}{n^2} < \frac{\pi/2}{n^2},$$
   and conclude that since $(\pi/2)\sum 1/n^2$ converges, $\sum \frac{\arctan(1/n)}{n^2}$ converges. Thus $\sum \frac{(-1)^{n-1} \arctan(1/n)}{n^2}$ is absolutely convergent.

7) In the following exercises use the limit comparison test to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{5n+1}{3n^2}$, by comparing to $\sum_{n=1}^{\infty} \frac{1}{n}$
   Answer: We have
   $$\frac{a_n}{b_n} = \frac{(5n+1)/(3n^2)}{1/n} = \frac{5n+1}{3n},$$
   so
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n+1}{3n} = \frac{5}{3} = c \neq 0.$$
   Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series, the original series diverges.

2. $\sum_{n=1}^{\infty} \left(\frac{1+n}{3n}\right)^n$, by comparing to $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$
   Answer: We have
   $$\frac{a_n}{b_n} = \frac{((1+n)/(3n))^n}{(1/3)^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n,$$
   so
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = c \neq 0.$$
   Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series, the original series converges.

3. $\sum \left(1 - \cos\frac{1}{n}\right)$, by comparing to $\sum 1/n^2$
   Answer: The n^{th} term is $a_n = 1 - \cos(1/n)$ and we are taking $b_n = 1/n^2$. We have
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2}.$$
   This limit is of the indeterminate form $0/0$ so we evaluate it using l'Hopital's rule. We have
   $$\lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2} = \lim_{n\to\infty} \frac{\sin(1/n)(-1/n^2)}{-2/n^3} = \lim_{n\to\infty} \frac{1}{2}\frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{1}{2}\frac{\sin x}{x} = \frac{1}{2}.$$
   The limit comparison test applies with $c = 1/2$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore $\sum(1-\cos(1/n))$ also converges.

4. $\sum \frac{1}{n^4-7}$
   Answer: The n^{th} term $a_n = 1/(n^4-7)$ behaves like $1/n^4$ for large n, so we take $b_n = 1/n^4$. We have
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^4-7)}{1/n^4} = \lim_{n\to\infty} \frac{n^4}{n^4-7} = 1.$$
   The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^4$ converges because $p = 4 > 1$. Therefore $\sum 1/(n^4-7)$ also converges.

5. $\sum \frac{n^3-2n^2+n+1}{n^4-2}$
   Answer: The n^{th} term $a_n = (n^3-2n^2+n+1)/(n^4-2)$ behaves like $n^3/n^4 = 1/n$ for large n, so we take $b_n = 1/n$. We have
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3-2n^2+n+1)/(n^4-2)}{1/n} = \lim_{n\to\infty} \frac{n^4-2n^3+n^2+n}{n^4-2} = 1.$$
   The limit comparison test applies with $c = 1$. The harmonic series $\sum 1/n$ diverges. Thus $\sum \left(n^3-2n^2+n+1\right) / \left(n^4-2\right)$ also diverges.

6. $\sum \frac{2^n}{3^n-1}$
   Answer: The n^{th} term $a_n = 2^n/(3^n-1)$ behaves like $2^n/3^n$ for large n, so we take $b_n = 2^n/3^n$. We have
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n/(3^n-1)}{2^n/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-1} = \lim_{n\to\infty} \frac{1}{1-3^{-n}} = 1.$$
   The limit comparison test applies with $c = 1$. The geometric series $\sum 2^n/3^n = \sum(2/3)^n$ converges. Therefore $\sum 2^n/(3^n-1)$ also converges.

7. $\sum \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$

Answer: The n^{th} term,
$$a_n = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{4n^2 - 2n},$$
behaves like $1/(4n^2)$ for large n, so we take $b_n = 1/(4n^2)$. We have
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(4n^2-2n)}{1/(4n^2)} = \lim_{n\to\infty} \frac{4n^2}{4n^2-2n} = \lim_{n\to\infty} \frac{1}{1-1/(2n)} = 1.$$
The limit comparison test applies with $c = 1$. The series $\sum 1/(4n^2)$ converges because it is a multiple of a p-series with $p = 2 > 1$. Therefore $\sum \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$ also converges.

8. $\sum \frac{1}{2\sqrt{n}+\sqrt{n+2}}$

Answer: The n^{th} term $a_n = 1/(2\sqrt{n} + \sqrt{n+2})$ behaves like $1/(3\sqrt{n})$ for large n, so we take $b_n = 1/(3\sqrt{n})$. We have
$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2\sqrt{n}+\sqrt{n+2})}{1/(3\sqrt{n})} &= \lim_{n\to\infty} \frac{3\sqrt{n}}{2\sqrt{n}+\sqrt{n+2}} \\ &= \lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}\left(2+\sqrt{1+2/n}\right)} \\ &= \lim_{n\to\infty} \frac{3}{2+\sqrt{1+2/n}} = \frac{3}{2+\sqrt{1+0}} \\ &= 1.\end{aligned}$$
The limit comparison test applies with $c = 1$. The series $\sum 1/(3\sqrt{n})$ diverges because it is a multiple of a p-series with $p = 1/2 < 1$. Therefore $\sum 1/(2\sqrt{n} + \sqrt{n+2})$ also diverges.

8) Explain why the integral test cannot be used to decide if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} n^2$

Answer: The integral test requires that $f(x) = x^2$, which is not decreasing.

2. $\displaystyle\sum_{n=1}^{\infty} e^{-n} \sin n$

Answer: The integral test requires that $f(x) = e^{-x} \sin x$, which is not positive, nor is it decreasing.

9) Explain why the comparison test cannot be used to decide if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Answer: The comparison test requires that $a_n = (-1)^n/n^2$ be positive. It is not.

2. $\displaystyle\sum_{n=1}^{\infty} \sin n$

Answer: The comparison test requires that $a_n = \sin n$ be positive for all n. It is not.

10) Explain why the ratio test cannot be used to decide if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

Answer: With $a_n = (-1)^n$, we have $|a_{n+1}/a_n| = 1$, and $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, so the test gives no information.

2. $\sum_{n=1}^{\infty} \sin n$

   Answer: With $a_n = \sin n$, we have $|a_{n+1}/a_n| = |\sin(n+1)/\sin n|$, which does not have a limit as $n \to \infty$, so the test does not apply.

11) Explain why the alternating series test cannot be used to decide if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} (-1)^{n-1} n$

   Answer: The sequence $a_n = n$ does not satisfy either $a_{n+1} < a_n$ or $\lim_{n\to\infty} a_n = 0$.

2. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(2 - \frac{1}{n}\right)$

   Answer: The alternating series test requires $a_n = 2 - 1/n$ which is positive and satisfies $a_{n+1} < a_n$ but $\lim_{n\to\infty} a_n = 2 \neq 0$.

12) JAMBALAYA!!! Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$

   Answer: We use the ratio test with $a_n = \frac{8^n}{n!}$. Replacing n by $n+1$ gives $a_{n+1} = \frac{8^{n+1}}{(n+1)!}$ and

   $$\frac{|a_{n+1}|}{|a_n|} = \frac{8^{n+1}/(n+1)!}{8^n/n!} = \frac{8n!}{(n+1)!} = \frac{8}{n+1}.$$

   Thus

   $$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{8}{n+1} = 0.$$

   Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{8^n}{n!}$ converges.

2. $\sum_{n=1}^{\infty} \frac{n2^n}{3^n}$

   Answer: We use the ratio test with $a_n = \frac{n2^n}{3^n}$. Replacing n by $n+1$ gives $a_{n+1} = \frac{(n+1)2^{n+1}}{3^{n+1}}$ and

   $$\frac{|a_{n+1}|}{|a_n|} = \frac{((n+1)2^{n+1})/3^{n+1}}{n2^n/3^n} = \frac{2(n+1)}{3n}.$$

   Thus

   $$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2(n+1)}{3n} = \lim_{n\to\infty} \frac{2(1+1/n)}{3} = \frac{2}{3}.$$

   Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{n2^n}{3^n}$ converges.

3. $\sum_{n=0}^{\infty} e^{-n}$

   Answer: The first few terms of the series may be written

   $$1 + e^{-1} + e^{-2} + e^{-3} + \cdots;$$

   this is a geometric series with $a = 1$ and $x = e^{-1} = 1/e$. Since $|x| < 1$, the geometric series converges to $S = \frac{1}{1-x} = \frac{1}{1-e^{-1}} = \frac{e}{e-1}$.

4. $\sum_{n=1}^{\infty} \frac{1}{n^2} \tan\left(\frac{1}{n}\right)$

   Answer: We compare the series with the convergent series $\sum 1/n^2$. From the graph of $\tan x$, we see that $\tan x < 2$ for $0 \le x \le 1$, so $\tan(1/n) < 2$ for all n. Thus

   $$\frac{1}{n^2} \tan\left(\frac{1}{n}\right) < \frac{1}{n^2} 2,$$

   so the series converges, since $2 \sum 1/n^2$ converges. Alternatively, we try the integral test. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding integral using the substitution $w = 1/x$:

   $$\int_1^{\infty} \frac{1}{x^2} \tan\left(\frac{1}{x}\right) dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2} \tan\left(\frac{1}{x}\right) dx = \lim_{b\to\infty} \ln\left(\cos\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty} \left(\ln\left(\cos\left(\frac{1}{b}\right)\right) - \ln(\cos 1)\right) = -\ln(\cos 1).$$

   Since the limit exists, the integral converges, so the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \tan(1/n)$ converges.

5. $\sum_{n=1}^{\infty} \frac{5n+2}{2n^2+3n+7}$

Answer: We use the limit comparison test with $a_n = \frac{5n+2}{2n^2+3n+7}$. Because a_n behaves like $\frac{5n}{2n^2} = \frac{5}{2n}$ as $n \to \infty$, we take $b_n = 1/n$.

We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(5n+2)}{2n^2+3n+7} = \frac{5}{2}.$$

By the limit comparison test (with $c = 5/2$) since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{5n+2}{2n^2+3n+7}$ also diverges.

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$

Answer: Let $a_n = 1/\sqrt{3n-1}$. Then replacing n by $n+1$ gives $a_{n+1} = 1/\sqrt{3(n+1)-1}$. Since

$$\sqrt{3(n+1)-1} > \sqrt{3n-1},$$

we have

$$a_{n+1} < a_n.$$

In addition, $\lim_{n\to\infty} a_n = 0$ so the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$ converges.

7. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Answer: Since $0 \le |\sin n| \le 1$ for all n, we may be able to compare with $1/n^2$. We have $0 \le |\sin n/n^2| \le 1/n^2$ for all n. So $\sum |\sin n/n^2|$ converges by comparison with the convergent series $\sum(1/n^2)$. Therefore $\sum(\sin n/n^2)$ also converges, since absolute convergence implies convergence.

8. $\sum_{n=2}^{\infty} \frac{3}{\ln n^2}$

Answer: Since

$$\frac{3}{\ln n^2} = \frac{3}{2\ln n},$$

our series behaves like the series $\sum 1/\ln n$. More precisely, for all $n \ge 2$, we have

$$0 \le \frac{1}{n} \le \frac{1}{\ln n} \le \frac{3}{2\ln n} = \frac{3}{\ln n^2},$$

so $\sum_{n=2}^{\infty} \frac{3}{\ln n^2}$ diverges by comparison with the divergent series $\sum \frac{1}{n}$.

9. $\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$ Answer: Let $a_n = n(n+1)/\sqrt{n^3+2n^2}$. Since $n^3+2n^2 = n^2(n+2)$, we have

$$a_n = \frac{n(n+1)}{n\sqrt{n+2}} = \frac{n+1}{\sqrt{n+2}}$$

so a_n grows without bound as $n \to \infty$, therefore the series $\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$ diverges.