Tests for Convergence of Series

1) Use the comparison test to confirm the statements in the following exercises.

1. \( \sum_{n=4}^{\infty} \frac{1}{n} \) diverges, so \( \sum_{n=4}^{\infty} \frac{1}{n-3} \) diverges.

Answer: Let \( a_n = 1/(n-3) \), for \( n \geq 4 \). Since \( n-3 < n \), we have \( 1/(n-3) > 1/n \), so

\[
a_n > \frac{1}{n}.
\]

The harmonic series \( \sum_{n=4}^{\infty} \frac{1}{n} \) diverges, so the comparison test tells us that the series \( \sum_{n=4}^{\infty} \frac{1}{n-3} \) also diverges.

2. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, so \( \sum_{n=1}^{\infty} \frac{1}{n^2+2} \) converges.

Answer: Let \( a_n = 1/(n^2+2) \). Since \( n^2+2 > n^2 \), we have \( 1/(n^2+2) < 1/n^2 \), so

\[
0 < a_n < \frac{1}{n^2}.
\]

The series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, so the comparison test tells us that the series \( \sum_{n=1}^{\infty} \frac{1}{n^2+2} \) also converges.

3. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, so \( \sum_{n=1}^{\infty} \frac{e^{-n}}{n^2} \) converges.

Answer: Let \( a_n = e^{-n}/n^2 \). Since \( e^{-n} < 1 \), for \( n \geq 1 \), we have \( \frac{e^{-n}}{n^2} < \frac{1}{n^2} \), so

\[
0 < a_n < \frac{1}{n^2}.
\]

The series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, so the comparison test tells us that the series \( \sum_{n=1}^{\infty} \frac{e^{-n}}{n^2} \) also converges.

2) Use the comparison test to determine whether the series in the following exercises converge.

1. \( \sum_{n=1}^{\infty} \frac{1}{3^n+1} \)

Answer: Let \( a_n = 1/(3^n+1) \). Since \( 3^n + 1 > 3^n \), we have \( 1/(3^n+1) < 1/3^n = (1/3)^n \), so

\[
0 < a_n < \left(\frac{1}{3}\right)^n.
\]

Thus we can compare the series \( \sum_{n=1}^{\infty} \frac{1}{3^n+1} \) with the geometric series \( \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \). This geometric series converges since \( |1/3| < 1 \), so the comparison test tells us that \( \sum_{n=1}^{\infty} \frac{1}{3^n+1} \) also converges.

2. \( \sum_{n=1}^{\infty} \frac{1}{n^4+e^n} \)

Answer: Let \( a_n = 1/(n^4+e^n) \). Since \( n^4 + e^n > n^4 \), we have

\[
\frac{1}{n^4+e^n} < \frac{1}{n^4},
\]

so

\[
0 < a_n < \frac{1}{n^4}.
\]

Since the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) converges, the comparison test tells us that the series \( \sum_{n=1}^{\infty} \frac{1}{n^4+e^n} \) also converges.

3. \( \sum_{n=2}^{\infty} \frac{1}{\ln n} \)

Answer: Since \( \ln n \leq n \) for \( n \geq 2 \), we have \( 1/\ln n \geq 1/n \), so the series diverges by comparison with the harmonic series, \( \sum 1/n \).
4. \( \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1} \)  
Answer: Let \( a_n = \frac{n^2}{n^4 + 1} \). Since \( n^4 + 1 > n^4 \), we have \( \frac{1}{n^4 + 1} < \frac{1}{n^4} \), so 
\[ a_n = \frac{n^2}{n^4 + 1} < \frac{n^2}{n^4} = \frac{1}{n^2}, \]
therefore 
\[ 0 < a_n < \frac{1}{n^2}. \]
Since the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, the comparison test tells us that the series \( \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1} \) converges also.

5. \( \sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^4 + 1} \)  
Answer: We know that \( n \sin^2 n \) is bounded, so 
\[ \frac{n \sin^2 n}{n^4 + 1} \leq \frac{n}{n^4 + 1} < \frac{n}{n^4} = \frac{1}{n^3}. \]
Since the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges, comparison gives that \( \sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^4 + 1} \) converges.

6. \( \sum_{n=1}^{\infty} \frac{2^n + 1}{n^2 \sin n} \)  
Answer: Let \( a_n = \frac{2^n + 1}{n^2 \sin n} \). Since \( 2^n - 1 < 2^n + n = n(2^n + 1) \), we have 
\[ \frac{2^n + 1}{2^n n^2} > \frac{2^n + 1}{n(2^n + 1)} = \frac{1}{n}. \]
Therefore, we can compare the series \( \sum_{n=1}^{\infty} \frac{2^n + 1}{n^2 \sin n} \) with the divergent harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \). The comparison test tells us that \( \sum_{n=1}^{\infty} \frac{2^n + 1}{n^2 \sin n} \) also diverges.

3) Use the ratio test to decide if the series in the following exercises converge or diverge.

1. \( \sum_{n=1}^{\infty} \frac{1}{(2n)!} \)  
Answer: Since \( a_n = 1/(2n)! \), replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = 1/(2n + 2)! \). Thus 
\[ \frac{|a_{n+1}|}{|a_n|} = \frac{(2n)!}{(2n + 2)!} = \frac{(2n)!}{(2n + 2)(2n + 1)(2n)!} = \frac{1}{(2n + 2)(2n + 1)}, \]
so 
\[ L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{1}{(2n + 2)(2n + 1)} = 0. \]
Since \( L = 0 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{1}{(2n)!} \) converges.

2. \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \)  
Answer: Since \( a_n = (n!)^2/(2n)! \), replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = ((n + 1)!)^2/(2n + 2)! \). Thus, 
\[ \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!^2(2n)!}{(2n + 2)(2n + 1)(2n)!} = \frac{(n+1)!^2}{(2n + 2)(2n + 1)} \cdot \frac{2n!}{(n!)^2}. \]
However, since \( (n + 1)! = (n + 1)n! \) and \( (2n + 2)! = (2n + 2)(2n + 1)(2n)! \), we have 
\[ \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2(n!)^2(2n)!}{(2n + 2)(2n + 1)(2n)!} = \frac{(n+1)^2}{(2n + 2)(2n + 1)} = \frac{n + 1}{4n + 2}, \]
so 
\[ L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}. \]
Since \( L < 1 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \) converges.
3. \( \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} \)

Answer: Since \( a_n = (2n)!/(n!(n+1)!) \), replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = (2n+2)!/((n+1)!(n+2)!) \). Thus,

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n+2)!}{(n+1)!(n+2)!} \cdot \frac{n!(n+1)!}{(2n)!}.
\]

However, since \( (n+2)! = (n+2)(n+1)n! \) and \( (2n+2)! = (2n+2)(2n+1)(2n)! \), we have

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n+2)(2n+1)}{(n+2)(n+1)} \cdot \frac{2n+1}{2n+2},
\]

so

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 4.
\]

Since \( L > 1 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} \) diverges.

4. \( \sum_{n=1}^{\infty} \frac{1}{r^n} \), \( r > 0 \)

Answer: Since \( a_n = 1/(r^n n!) \), replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = 1/(r^{n+1}(n+1)!) \). Thus

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{r} \frac{n+1}{n} \cdot \frac{1}{r} \frac{n}{n-1} = \frac{1}{r^{n+1}(n+1)!
\]

so

\[
L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{r} \lim_{n \to \infty} \frac{1}{n+1} = 0.
\]

Since \( L = 0 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{1}{r^n} \) converges for all \( r > 0 \).

5. \( \sum_{n=1}^{\infty} \frac{1}{n e^n} \)

Answer: Since \( a_n = 1/(n e^n) \), replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = 1/(n+1)e^{n+1} \). Thus

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{n+1} \frac{1}{e^{n+1}} = \frac{ne^n}{(n+1)e^{n+1}} = \left( \frac{n}{n+1} \right) \frac{1}{e}.
\]

Therefore

\[
L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1.
\]

Since \( L < 1 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{1}{n e^n} \) converges.

6. \( \sum_{n=0}^{\infty} \frac{2^n}{n^3+1} \)

Answer: Since \( a_n = 2^n/(n^3 + 1) \), replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = 2^{n+1}/((n+1)^3 + 1) \). Thus

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)^3 + 1} \cdot \frac{n^3 + 1}{2^n} = \frac{2^{n+1}}{n^3 + 1} \cdot \frac{n^3 + 1}{2^n} = \frac{2^{n+1}}{(n+1)^3 + 1},
\]

so

\[
L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 2.
\]

Since \( L > 1 \) the ratio test tells us that the series \( \sum_{n=0}^{\infty} \frac{2^n}{n^3+1} \) diverges.

4) Use the integral test to decide whether the following series converge or diverge.

1. \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

Answer: We use the integral test with \( f(x) = 1/x^3 \) to determine whether this series converges or diverges.

We determine whether the corresponding improper integral \( \int_1^{\infty} \frac{1}{x^3} dx \) converges or diverges:

\[
\int_1^{\infty} \frac{1}{x^3} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^3} dx = \lim_{b \to \infty} \left[ -\frac{1}{2x^2} \right]_1^b = \lim_{b \to \infty} \left( -\frac{1}{2b^2} + \frac{1}{2} \right) = \frac{1}{2}.
\]

Since the integral \( \int_1^{\infty} \frac{1}{x^3} dx \) converges, we conclude from the integral test that the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges.
2. \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)

Answer: We use the integral test with \( f(x) = \frac{x}{x^2 + 1} \) to determine whether this series converges or diverges.

We determine whether the corresponding improper integral \( \int_{1}^{\infty} \frac{x}{x^2 + 1} \, dx \) converges or diverges:

\[
\int_{1}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \frac{1}{2} \ln(x^2 + 1) \bigg|_{1}^{b} = \lim_{b \to \infty} \left( \frac{1}{2} \ln(b^2 + 1) - \frac{1}{2} \ln 2 \right) = \infty.
\]

Since the integral \( \int_{1}^{\infty} \frac{x}{x^2 + 1} \, dx \) diverges, we conclude from the integral test that the series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) diverges.

3. \( \sum_{n=1}^{\infty} \frac{1}{e^n} \)

Answer: We use the integral test with \( f(x) = \frac{1}{e^x} \) to determine whether this series converges or diverges.

To do so we determine whether the corresponding improper integral \( \int_{1}^{\infty} \frac{1}{e^x} \, dx \) converges or diverges:

\[
\int_{1}^{\infty} \frac{1}{e^x} \, dx = \lim_{b \to \infty} \int_{1}^{b} e^{-x} \, dx = \lim_{b \to \infty} -e^{-x} \bigg|_{1}^{b} = \lim_{b \to \infty} (-e^{-b} + e^{-1}) = e^{-1}.
\]

Since the integral \( \int_{1}^{\infty} \frac{1}{e^x} \, dx \) converges, we conclude from the integral test that the series \( \sum_{n=1}^{\infty} \frac{1}{e^n} \) converges.

We can also observe that this is a geometric series with ratio \( x = \frac{1}{e} < 1 \), and hence it converges.

4. \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \)

Answer: We use the integral test with \( f(x) = \frac{1}{x(\ln x)^2} \) to determine whether this series converges or diverges. We determine whether the corresponding improper integral \( \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx \) converges or diverges:

\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x(\ln x)^2} \, dx = \lim_{b \to \infty} -\frac{1}{\ln x} \bigg|_{2}^{b} = \lim_{b \to \infty} \left( -\frac{1}{\ln b} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}.
\]

Since the integral \( \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx \) converges, we conclude from the integral test that the series \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges.

5) Use the alternating series test to show that the following series converge.

1. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \)

Answer: Let \( a_n = 1/\sqrt{n} \). Then replacing \( n \) by \( n + 1 \) we have \( a_{n+1} = 1/\sqrt{n + 1} \). Since \( \sqrt{n + 1} > \sqrt{n} \), we have \( \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \), hence \( a_{n+1} < a_n \). In addition, \( \lim_{n \to \infty} a_n = 0 \) so \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges by the alternating series test.

2. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} \)

Answer: Let \( a_n = 1/(2n + 1) \). Then replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = 1/(2n + 3) \). Since \( 2n + 3 > 2n + 1 \), we have

\[
0 < a_{n+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = a_n.
\]

We also have \( \lim_{n \to \infty} a_n = 0 \). Therefore, the alternating series test tells us that the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} \) converges.

3. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 2n + 1} \)

Answer: Let \( a_n = 1/(n^2 + 2n + 1) = 1/(n + 1)^2 \). Then replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = 1/(n + 2)^2 \). Since \( n + 2 > n + 1 \), we have

\[
\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2}
\]
We also have \( \lim_{n \to \infty} a_n = 0 \). Therefore, the alternating series test tells us that the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 2n + 1} \) converges.

4. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n} \)
Answer: Let \( a_n = 1/e^n \). Then replacing \( n \) by \( n + 1 \) we have \( a_{n+1} = 1/e^{n+1} \). Since \( e^{n+1} > e^n \), we have \( \frac{1}{e^{n+1}} < \frac{1}{e^n} \), hence \( a_{n+1} < a_n \). In addition, \( \lim_{n \to \infty} a_n = 0 \) so \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n} \) converges by the alternating series test. We can also observe that the series is geometric with ratio \( x = -1/e \) can hence converges since \( |x| < 1 \).

6) In the following exercises determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. \( \sum \frac{(-1)^n}{2^n} \)
Answer: Both \( \sum \frac{(-1)^n}{2^n} = \sum \left( \frac{-1}{2} \right)^n \) and \( \sum \frac{1}{2^n} = \sum \left( \frac{1}{2} \right)^n \) are convergent geometric series. Thus \( \sum \frac{(-1)^n}{2^n} \) is absolutely convergent.

2. \( \sum \frac{(-1)^n}{n^2 + 7} \)
Answer: The series \( \sum \frac{(-1)^n}{n^2 + 7} \) converges by the alternating series test. However \( \sum \frac{1}{2n} \) diverges because it is a multiple of the harmonic series. Thus \( \sum \frac{(-1)^n}{n^2 + 7} \) is conditionally convergent.

3. \( \sum (-1)^n \left( 1 + \frac{1}{n^2} \right) \)
Answer: Since \( \lim_{n \to \infty} \left( 1 + \frac{1}{n^2} \right) = 1 \), the \( n \)th term \( a_n = (-1)^n \left( 1 + \frac{1}{n^2} \right) \) does not tend to zero as \( n \to \infty \). Thus, the series \( \sum (-1)^n \left( 1 + \frac{1}{n^2} \right) \) is divergent.

4. \( \sum \frac{(-1)^n}{n^2 + 1} \)
Answer: The series \( \sum \frac{(-1)^n}{n^2 + 1} \) converges by the alternating series test. Moreover, the series \( \sum \frac{1}{n^2 + 1} \) converges by comparison with the convergent \( p \)-series \( \sum \frac{1}{n^2} \). Thus \( \sum \frac{(-1)^n}{n^2 + 1} \) is absolutely convergent.

5. \( \sum \frac{(-1)^{n-1}}{n \ln n} \)
Answer: We first check absolute convergence by deciding whether \( \sum 1/(n \ln n) \) converges by using the integral test. Since
\[
\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^{b} \frac{dx}{x \ln x} = \lim_{b \to \infty} \ln \ln(b) - \ln \ln(2)),
\]
and since this limit does not exist, \( \sum \frac{1}{n \ln n} \) diverges.

We now check conditional convergence. The original series is alternating so we check whether \( a_{n+1} < a_n \). Consider \( a_n = f(n) \), where \( f(x) = 1/(x \ln x) \). Since
\[
\frac{d}{dx} \left( \frac{1}{x \ln x} \right) = -\frac{1}{x^2 \ln x} \left( 1 + \frac{1}{\ln x} \right)
\]
is negative for \( x > 1 \), we know that \( a_n \) is decreasing for \( n \geq 2 \). Thus, for \( n \geq 2 \)
\[
a_{n+1} = \frac{1}{(n+1) \ln(n+1)} < -\frac{1}{n \ln n} = a_n.
\]
Since \( 1/(n \ln n) \to 0 \) as \( n \to \infty \), we see that \( \sum \frac{(-1)^{n-1}}{n \ln n} \) is conditionally convergent.

6. \( \sum \frac{(-1)^{n-1} \arctan(1/n)}{n^2} \)
Answer: We first check absolute convergence by deciding whether \( \sum \frac{\arctan(1/n)}{n^2} \) converges. Since \( \arctan x \) is the angle between \(-\pi/2 \) and \( \pi/2 \), we have \( \arctan(1/n) < \pi/2 \) for all \( n \). We compare
\[
\frac{\arctan(1/n)}{n^2} < \frac{\pi/2}{n^2},
\]
and conclude that since \( \pi/2 \sum 1/n^2 \) converges, \( \sum \frac{\arctan(1/n)}{n^2} \) converges. Thus \( \sum \frac{(-1)^{n-1} \arctan(1/n)}{n^2} \) is absolutely convergent.
In the following exercises use the limit comparison test to determine whether the series converges or diverges.

1. \( \sum_{n=1}^{\infty} \frac{5n+1}{3n^2} \), by comparing to \( \sum_{n=1}^{\infty} \frac{1}{n} \)

   Answer: We have 
   \[
   \frac{a_n}{b_n} = \frac{(5n+1)/(3n^2)}{1/n} = \frac{5n+1}{3n},
   \]
   so 
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{5n+1}{3n} = \frac{5}{3} = c \neq 0.
   \]
   Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is a divergent harmonic series, the original series diverges.

2. \( \sum_{n=1}^{\infty} \left( \frac{1+n}{3n} \right)^n \), by comparing to \( \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \)

   Answer: We have 
   \[
   \frac{a_n}{b_n} = \frac{(1+n)/(3n)}{(1/3)^n} = \left( \frac{n+1}{n} \right)^n = \left( 1 + \frac{1}{n} \right)^n,
   \]
   so 
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e = c \neq 0.
   \]
   Since \( \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \) is a convergent geometric series, the original series converges.

3. \( \sum (1 - \cos \frac{1}{n}) \), by comparing to \( \sum 1/n^2 \)

   Answer: The \( n^{th} \) term is \( a_n = 1 - \cos(1/n) \) and we are taking \( b_n = 1/n^2 \). We have 
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1 - \cos(1/n)}{1/n^2}.
   \]
   This limit is of the indeterminate form 0/0 so we evaluate it using l’Hopital’s rule. We have 
   \[
   \lim_{n \to \infty} \frac{1 - \cos(1/n)}{1/n^2} = \lim_{n \to \infty} \frac{\sin(1/n)(-1/n^2)}{-2/n^3} = \lim_{n \to \infty} \frac{1 \sin(1/n)}{2 \cdot 1/n} = \lim_{x \to 0} \frac{1 \sin x}{2 \cdot x} = \frac{1}{2}.
   \]
   The limit comparison test applies with \( c = 1/2 \). The \( p \)-series \( \sum 1/n^2 \) converges because \( p = 2 > 1 \). Therefore \( \sum (1 - \cos(1/n)) \) also converges.

4. \( \sum \frac{1}{n^{1/4}} \)

   Answer: The \( n^{th} \) term \( a_n = 1/(n^4 - 7) \) behaves like \( 1/n^4 \) for large \( n \), so we take \( b_n = 1/n^4 \). We have 
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(n^4 - 7)}{1/n^4} = \lim_{n \to \infty} \frac{n^4}{n^4 - 7} = 1.
   \]
   The limit comparison test applies with \( c = 1 \). The \( p \)-series \( \sum 1/n^4 \) converges because \( p = 4 > 1 \). Therefore \( \sum 1/(n^4 - 7) \) also converges.

5. \( \sum \frac{n^3 - 2n^2 + n + 1}{n^2 - 2} \)

   Answer: The \( n^{th} \) term \( a_n = (n^3 - 2n^2 + n + 1)/(n^4 - 2) \) behaves like \( n^3/n^4 = 1/n \) for large \( n \), so we take \( b_n = 1/n \). We have 
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(n^3 - 2n^2 + n + 1)/(n^4 - 2)}{1/n} = \lim_{n \to \infty} \frac{n^4 - 2n^3 + n^2 + n}{n^4 - 2} = 1.
   \]
   The limit comparison test applies with \( c = 1 \). The harmonic series \( \sum 1/n \) diverges. Thus \( \sum (n^3 - 2n^2 + n + 1) / (n^4 - 2) \) also diverges.

6. \( \sum \frac{2^n}{3^n - 1} \)

   Answer: The \( n^{th} \) term \( a_n = 2^n/(3^n - 1) \) behaves like \( 2^n/3^n \) for large \( n \), so we take \( b_n = 2^n/3^n \). We have 
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2^n/(3^n - 1)}{2^n/3^n} = \lim_{n \to \infty} \frac{3^n}{3^n - 1} = \lim_{n \to \infty} \frac{1}{1 - 3^{-n}} = 1.
   \]
   The limit comparison test applies with \( c = 1 \). The geometric series \( \sum 2^n/3^n = \sum (2/3)^n \) converges. Therefore \( \sum 2^n/(3^n - 1) \) also converges.
7. \[ \sum \left( \frac{1}{2n-1} - \frac{1}{2n} \right) \]
Answer: The \( n \)th term, 
\[ a_n = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{4n^2 - 2n}, \]
behaves like \( 1/(4n^2) \) for large \( n \), so we take \( b_n = 1/(4n^2) \). We have
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(4n^2 - 2n)}{1/(4n^2)} = \lim_{n \to \infty} \frac{4n^2}{4n^2 - 2n} = \lim_{n \to \infty} \frac{1}{1 - 1/(2n)} = 1. \]
The limit comparison test applies with \( c = 1 \). The series \( \sum 1/(4n^2) \) converges because it is a multiple of a \( p \)-series with \( p = 2 > 1 \). Therefore \( \sum \left( \frac{1}{2n-1} - \frac{1}{2n} \right) \) also converges.

8. \[ \sum \frac{1}{2\sqrt{n} + \sqrt{n+2}} \]
Answer: The \( n \)th term \( a_n = 1/(2\sqrt{n} + \sqrt{n+2}) \) behaves like \( 1/(3\sqrt{n}) \) for large \( n \), so we take \( b_n = 1/(3\sqrt{n}) \). We have
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(2\sqrt{n} + \sqrt{n+2})}{1/(3\sqrt{n})} = \lim_{n \to \infty} \frac{3\sqrt{n}}{2\sqrt{n} + \sqrt{n+2}} = \lim_{n \to \infty} \frac{3}{2 + \sqrt{1 + 2/n}} = \frac{3}{2 + \sqrt{1 + 0}} = 1. \]
The limit comparison test applies with \( c = 1 \). The series \( \sum 1/(3\sqrt{n}) \) diverges because it is a multiple of a \( p \)-series with \( p = 1/2 < 1 \). Therefore \( \sum 1/(2\sqrt{n} + \sqrt{n+2}) \) also diverges.

8) Explain why the integral test cannot be used to decide if the following series converge or diverge.

1. \[ \sum_{n=1}^{\infty} n^2 \]
Answer: The integral test requires that \( f(x) = x^2 \), which is not decreasing.

2. \[ \sum_{n=1}^{\infty} e^{-n} \sin n \]
Answer: The integral test requires that \( f(x) = e^{-x} \sin x \), which is not positive, nor is it decreasing.

9) Explain why the comparison test cannot be used to decide if the following series converge or diverge.

1. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \]
Answer: The comparison test requires that \( a_n = (-1)^n/n^2 \) be positive. It is not.

2. \[ \sum_{n=1}^{\infty} \sin n \]
Answer: The comparison test requires that \( a_n = \sin n \) be positive for all \( n \). It is not.

10) Explain why the ratio test cannot be used to decide if the following series converge or diverge.

1. \[ \sum_{n=1}^{\infty} (-1)^n \]
Answer: With \( a_n = (-1)^n \), we have \( |a_{n+1}/a_n| = 1 \), and \( \lim_{n \to \infty} |a_{n+1}/a_n| = 1 \), so the test gives no information.
2. \( \sum_{n=1}^{\infty} \sin n \)
   Answer: With \( a_n = \sin n \), we have \( |a_{n+1}/a_n| = |\sin(n+1)/\sin n| \), which does not have a limit as \( n \to \infty \), so the test does not apply.

11) Explain why the alternating series test cannot be used to decide if the following series converge or diverge.

1. \( \sum_{n=1}^{\infty} (-1)^{n-1} n \)
   Answer: The sequence \( a_n = n \) does not satisfy either \( a_{n+1} < a_n \) or \( \lim_{n \to \infty} a_n = 0 \).

2. \( \sum_{n=1}^{\infty} (-1)^{n-1} \left( 2 - \frac{1}{n} \right) \)
   Answer: The alternating series test requires \( a_n = 2 - 1/n \) which is positive and satisfies \( a_{n+1} < a_n \) but \( \lim_{n \to \infty} a_n = 2 \neq 0 \).

12) JAMBALAYA!!! Determine if the following series converge or diverge.

1. \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \)
   Answer: We use the ratio test with \( a_n = \frac{2^n}{n!} \). Replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = \frac{2^{n+1}}{(n+1)!} \) and
   \[
   \frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{n!} \cdot \frac{n!}{(n+1)!} = \frac{2(n+1)}{n+1}.\]
   Thus
   \[
   L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{2(n+1)}{n+1} = 2.
   \]
   Since \( L > 1 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \) diverges.

2. \( \sum_{n=1}^{\infty} \frac{2^n}{3^n} \)
   Answer: We use the ratio test with \( a_n = \frac{2^n}{3^n} \). Replacing \( n \) by \( n + 1 \) gives \( a_{n+1} = \frac{(n+1)2^{n+1}}{3^{n+1}} \) and
   \[
   \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)2^{n+1}}{3^{n+1}} \cdot \frac{3^n}{2^n} = \frac{2(n+1)}{3}.\]
   Thus
   \[
   L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{2(n+1)}{3} = \frac{2}{3}.
   \]
   Since \( L < 1 \), the ratio test tells us that \( \sum_{n=1}^{\infty} \frac{2^n}{3^n} \) converges.

3. \( \sum_{n=0}^{\infty} e^{-n} \)
   Answer: The first few terms of the series may be written
   \[
   1 + e^{-1} + e^{-2} + e^{-3} + \cdots ;
   \]
   this is a geometric series with \( a = 1 \) and \( x = e^{-1} = 1/e \). Since \( |x| < 1 \), the geometric series converges to
   \[
   S = \frac{1}{1-x} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}.
   \]

4. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \tan \left( \frac{1}{n} \right) \)
   Answer: We compare the series with the convergent series \( \sum 1/n^2 \). From the graph of \( \tan x \), we see that \( \tan x < 2 \) for \( 0 \leq x \leq 1 \), so \( \tan(1/n) < 2 \) for all \( n \). Thus
   \[
   \frac{1}{n^2} \tan \left( \frac{1}{n} \right) < \frac{1}{n^2} 2,
   \]
   so the series converges, since \( 2 \sum 1/n^2 \) converges. Alternatively, we try the integral test. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding integral using the substitution \( w = 1/x \):
   \[
   \int_{1}^{\infty} \frac{1}{x^2} \tan \left( \frac{1}{x} \right) dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2} \tan \left( \frac{1}{x} \right) dx = \lim_{b \to \infty} \ln \left( \cos \left( \frac{1}{x} \right) \right) \bigg|_{1}^{b} = \lim_{b \to \infty} \left( \ln \left( \cos \left( \frac{1}{b} \right) \right) - \ln(\cos 1) \right) = -\ln(\cos 1).
   \]
   Since the limit exists, the integral converges, so the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \tan \left( \frac{1}{n} \right) \) converges.
5. \( \sum_{n=1}^{\infty} \frac{5n+2}{2n^2+3n+7} \)

Answer: We use the limit comparison test with \( a_n = \frac{5n+2}{2n^2+3n+7} \). Because \( a_n \) behaves like \( \frac{5n}{2n^2} = \frac{5}{2n} \) as \( n \to \infty \), we take \( b_n = \frac{1}{n} \).

We have

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n(5n + 2)}{2n^2 + 3n + 7} = \frac{5}{2}.
\]

By the limit comparison test (with \( c = \frac{5}{2} \)) since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, \( \sum_{n=1}^{\infty} \frac{5n+2}{2n^2+3n+7} \) also diverges.

6. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{3n-1}} \)

Answer: Let \( a_n = \frac{1}{\sqrt[3]{3n-1}} \). Then replacing \( n \) by \( n+1 \) gives \( a_{n+1} = \frac{1}{\sqrt[3]{3(n+1)-1}} \).

We have

\[
\sqrt[3]{3(n+1)-1} > \sqrt[3]{3n-1},
\]

so \( a_{n+1} < a_n \).

In addition, \( \lim_{n \to \infty} a_n = 0 \) so the alternating series test tells us that the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{3n-1}} \) converges.

7. \( \sum_{n=1}^{\infty} \frac{\sin n}{\pi} \)

Answer: Since \( 0 \leq |\sin n| \leq 1 \) for all \( n \), we may be able to compare with \( \frac{1}{n^2} \). We have \( 0 \leq |\sin n/n^2| \leq \frac{1}{n^2} \) for all \( n \). So \( \sum |\sin n/n^2| \) converges by comparison with the convergent series \( \sum (1/n^2) \). Therefore \( \sum (\sin n/n^2) \) also converges, since absolute convergence implies convergence.

8. \( \sum_{n=2}^{\infty} \frac{3}{\ln n^2} \)

Answer: Since

\[
\frac{3}{\ln n^2} = \frac{3}{2\ln n},
\]

our series behaves like the series \( \sum 1/\ln n \). More precisely, for all \( n \geq 2 \), we have

\[
0 \leq \frac{1}{n} \leq \frac{1}{\ln n} \leq \frac{3}{2\ln n} = \frac{3}{\ln n^2},
\]

so \( \sum_{n=2}^{\infty} \frac{3}{\ln n^2} \) diverges by comparison with the divergent series \( \sum \frac{1}{n} \).

9. \( \sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}} \)

Answer: Let \( a_n = \frac{n(n+1)}{\sqrt{n^3+2n^2}} \). Since \( n^3 + 2n^2 = n^2(n+2) \), we have

\[
a_n = \frac{n(n+1)}{n\sqrt{n+2}} = \frac{n+1}{\sqrt{n+2}}
\]

so \( a_n \) grows without bound as \( n \to \infty \), therefore the series \( \sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}} \) diverges.