Selections from Ibn al-Haytham (965-1041): Treatise on the volume of the sphere.

Translation by Jan P. Hogendijk based on the Arabic edition by R. Rashed. References such as [295] are to the pages of the Arabic edition; see the bibliography at the end. Explanatory additions by Jan P. Hogendijk are in square brackets.

[295] In the name of God, the Merciiful, the Compassionate.

Treatise by al-Ḥasan ibn al-Ḥasan ibn al-Haytham on the volume of the sphere.

Many things in geometry can be obtained by different approaches, and the proof of them is possible in a number of ways.Thus, one mathematician can continue to work on a a matter that had been discussed by others before, and he can reach the goal, even if someone else had arrived at it before, if he finds a (new) method to discuss it which had not been found by any earlier (mathematician) before him. A number of mathematicians have talked about the measurement of the sphere, and they have established the proof for the quantity of its volume. Each of them followed a method different from the others.

Since their discussions of this matter have reached us and we have become acquainted with their proofs, we have thought deeply about the volume of the sphere - whether it can be obtained by a method different from the methods that had been used by those that have discussed the subject previously. When we spent a lot of attention on this, a new way occurred to us for finding the volume of the sphere which is more concise and shorter than all the ways which our predecessors have followed, with a clearer proof and a more evident explanation. So in this situation it is permissible for us to discuss the volume of the sphere, although a number of mathematicians have already talked about it before us.

[297] We begin this with an easy aritmetical lemma which will facilitate the understanding of our aim. It is as follows. If we consider [a finite series of] the successive integers, beginning with one, and increasing by one at a time, and if one-third of the greatest number is added to one-third, and the sum is multiplied by the greatest number, and one half is added to the greatest number, and this (sum) is multiplied by the first product, then

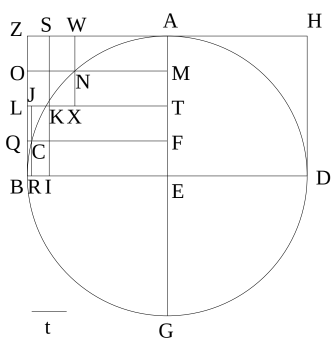
the result is the sum of the squares of these numbers. [in modern terms $1^2 + 2^2 + \ldots + n^2 = (\frac{1}{3}n + \frac{1}{3})n(n + \frac{1}{2})$.]

...[299] ...[301]

[Ibn al-Haytham then draws an easy consequence, which can be stated in modern terms as follows: $1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n^3 + c$ with $\frac{1}{2}n^2 < c < \frac{2}{3}n^2$.]

[307] We say: Every sphere is two thirds of the cylinder whose base is the greatest circle in the sphere and whose height is equal to the diameter of the sphere.

The example of this is the sphere $ABGD$ with centre E. Then I say: this is two thirds of the cylinder whose base is the greatest circle in the sphere and whose height is the diameter of the sphere.



(Proof): We let pass through the center E of the sphere a plane which will intersect the sphere in a great circle $ABGD$. In this circle we draw two perpendicular diameters AEG and BED. Through B we draw the line BZ parallel to line EA. Through point A we draw the line AZ parallel to line EB, so $AEBZ$ is a rectangle. If line AE is fixed and rectangle $AEBZ$ is turned around line AE until it has returned to its original position, rectangle $AEBZ$ produces a cylinder whose base is the circle with radius line BE, which is a radius of the sphere, and height line EA which is also a radius of the sphere. A circle [309] whose radius is a radius of the sphere is a great circle on the sphere. So the cylinder which is produced by the rotation of rectangle BA around line EA has as its basis a great circle of the sphere and height the radius of the sphere. Let this cylinder be BH. If rectangle BA rotates around line EA, sector ABE also rotates around line EA, and if sector ABE rotates around line EA, the rotation produces a hemisphere whose base is the circle with radius line BE.

For if half of the circle $ABGD$, containing ABG and the diameter AG, rotates around the diameter AG until it returns to its original position, the rotation produces the sphere $ABGD$, and the rotation of line EB produces a circle which bisects the sphere. So if the rectangle BA turns around the line EA, the rotation produces a cylinder with basis a great circle on the sphere $ABGD$, and height line EA, which is a radius of the sphere $ABGD$. (Thus) the rotation of sector ABE produces half of the sphere $ABGD$.

Then I say: the hemisphere produced by the rotation of sector ABE is (equal to) two thirds of the cylinder BH which is produced by the rotation of rectangle BA.

Proof of this: it cannot be otherwise. For if it were possible, let the hemisphere be unequal to two thirds of the cylinder BH. Since the hemisphere is unequal to two thirds of the cylinder BH, it is either greater than two thirds of the cylinder or less.

So let the hemisphere first be greater than two thirds of the cylinder, and let the hemisphere exceed two thirds of the cylinder by the magnitude t.

We bisect AE at point T, and we draw through point T line TK parallel to line EB. Then TK is perpendicular to line AE. We extend TK to L, then TL is equal to line EB. We draw through point K a line SKI parallel to the two lines EA, BZ. Then SK is equal to KI, since AT is equal to TE. So rectangle KE is equal to rectangle KA, and rectangle KB is equal to rectangle KZ. If rectangle BA rotates around line EA, the rectangles EK, KA produce two [311] equal cylinders, and the two figures KB, KZ produce two equal round solids which circumscribe the equal cylinders. So the cylinder produced by the rotation of rectangle KE and the round solid produced by the rotation of rectangle KZ are together half of the cylinder BH

Again, we bisect AT at the point M, and we draw through point M line MN parallel to line EB. Then MN is perpendicular to line AE. We extend MN to O. Then MO is equal to line EB. We draw through point N line WNX parallel to the lines KS, LZ. Then WN is equal to NX, and rectangle NT is equal to rectangle NA, and rectangle NK is equal to rectangle NS. If rectangle BA rotates around line EA, rectangle KA rotates (also). Then the rectangles NT, NA produce two equal cylinders, and the rectangles NK, NS produce two round solids. The cylinder produced by the rotation of rectangle NT together with the round solid produced by the rotation of rectangle NS are together half of the cylinder which is produced by the rotation of rectangle KA.

Again we bisect line TE at point F. We draw through F line FC parallel to line EB. Then FC is perpendicular to AE. We extend FC to Q. Then FQ is equal to line EB. We draw through point C line JCR^* parallel to the lines ET, BL. Then JC is equal to CR^*, and rectangle CK is equal to rectangle CI, and rectangle CB is equal to rectangle CL. If rectangle BA rotates around line EA, rectangle BK (also rotates), and the rotation of it produces a round solid, and the rotation of rectangles CK, CI produces two round solids, and the rotation of rectangles CB, CL produces two round solids. Then the round solid produced by the [313] rotation of rectangle CI together with the round solid produced by the rotation of rectangle CL, equal to half of the round solid produced by the rotation of rectangle KB.

So the (sum of the) cylinder produced by the rotation of rectangle NT together with the round solid produced by the rotation of rectangle NS together with the two round solids produced by the rotation of the two rectangles CI, CL is half of the cylinder produced by the rotation of rectangle KA together with half of the round solid produced by the rotation of rectangle KB. It has been proved that the (sum of the) cylinder produced by the rotation of rectangle KE together with the round solid produced by the rotation of rectangle KZ is equal to half of cylinder BH.

Since this is the case, half of the cylinder BH has been removed, and half of the remainder has also been removed. If we bisect all lines AM, MT, TF, FE, and from the points of bisection we draw lines parallel to line BE, and if from their intersection points with arc AB we draw lines parallel to line AE, the rectangles BC, CK, KN, NA are all divided into four parts and (of these four parts,) each pair of opposite rectangles is equal to half of the rectangle out of which they were produced, and the round solids which are produced by the rotation of these two rectangles are half (the volume) of the round solids which are produced by the rotation of rectangles BC, CK, KN, NA. If this (process) is always continued, half of the cylinder BH is cut off, and half of the remainder, and half of the (new) remainder. . . . (the text is unclear, probably the manuscripts are corrupted, and the editor Rashed does not understand the mathematics)

For each pair of different magnitudes, the following holds: if half of the greater of them is cut off, and half of the remainder, and if this is always continued, then it is necessary that (eventually) a magnitude will remain which is less than the lesser of the two (original) magnitudes. For if half of the (first) magnitude is cut off, and half of the remainder is cut off, (so we have been cutting) two times, more than half of the (first) magnitude has been cut off. So if half of a magnitude is cut off, and then half of the remainder, and if we do this many times, we will cut off in every two steps more than half. For each pair of different magnitudes the following (theorem) holds: if of the greater $<$ more than $>$ half is cut off, and from the remainder $<$ more than $>$ half, and if this (operation) is continued indefinitely, then eventually a magnitude will remain which is less than the lesser magnitude (of the two). [This theorem was proved by Euclid iin Book X of the Elements]

The cylinder BH and the magnitude t are two different magnitudes, and the greater of them is the cylinder BH. So if half of the cylinder BH is cut off, and half of the remainder, and half of the remainder, in the way which we have explained, and if this is always continued, then (eventually) there will remain a magnitude less than the magnitude t. [315] And if half of the cylinder BH is cut off, and half of the remainder, and half of the remainder, in the way which we have explained, then what remains of the cylinder consists of the round solids produced by the rectangles BC, CK, KN, NA and corresponding ones such that the surface of the sphere passes through their interiors.

Now let the parts at which the division of the cylinder as we have explained ends, and they are the parts which (together) are less than t, be the round solids which are produced by the rotation of the rectangles BC, CK, KN, NA. Then the parts of these round solids which lie inside the sphere are much less than the magnitude t. But the hemisphere exceeds two thirds of the cylinder by the magnitude t. So the remainder of the hemisphere, after these parts of the round solids in its interior have been removed, is greater than two thirds of the cylinder BH. But the remainder of the hemisphere after the parts of the round solids in the interior of it (the sphere) have been removed, is the sawed-off solid in the interior of the hemisphere, whose base is the circle with radius RE*, and whose top is the circle with radius MN. So this sawed-off solid is greater than two thirds of the cylinder BH.

Again: the lines AM, MT, TF, FE are equal, so each of the lines EF, ET, EM, EA exceeds its predecessor by line EF. So the ratios of the lines EF, ET, EM, EA are the ratios of the successive (natural) numbers, beginning with one, and always increasing by one.

So the squares of the lines EF, ET, EM, EA exceed one-third of the equal squares which are equal to the square of EA, and whose number is the number of lines EF, ET, EM, EA; and the excess is less than two-thirds of the square of EA, as has been proved in the lemma [which I have not yet translated]

The number of lines EF, ET, EM, EA is the number of division points

F, T, M, A, and the number of division points F, T, M, A is the number of division points E, F, T, M if we take E instead of A. But the number of division points E, F, T, M is equal to the number of lines EB, FQ, TL, MO. The lines EB, FQ, TL, MO are equal to one another and each of them is equal to line EB, and EB is equal to line EA. So the squares of the lines EF, ET, EM, EA exceed one-third of the squares of the lines EB, FQ, TL, MO by less than two thirds of the square of EA. The square of EF [317] together with the product GF by FA is the square of EA. [In modern terms $a^2 - b^2 = (a + b)(a - b)$ with $EA = EG = a$ and $EF = b$.] But the product GF by FA is the square of FC. [This is a property of the circle.] So the square of EF together with the square of FC is equal to the square of EA [he could have concluded this directly from the Theorem of Pythagoras. Has the text been changed by a later scribe?], which is equal to the square of FQ. In the same way, the square of ET together with the square of TK is equal to the square of EA, which is equal is to the square of TL. In the same way, the square of EM together with the square of MN is equal to the square of EA, which is equal to the square of MO. And the square of EA is equal to the square of EB. So the squares of EF, ET, EM, EA together with the squares of FC, TK, MN are together equal to the squares of EB, FQ, TL, MO. But the squares of EF, ET, EM, EA exceed one third of the squares of EB, FQ, TL, MO by less than two thirds of the square of EA. So, by subtraction, the squares FC, TK, MN are less than two thirds of the squares of EB, FQ, TL, MO, and the difference is less than two thirds of the square of EA.

So the circles whose radii are the lines FC, TK, MN are less than two thirds times the circles whose radii are EB, FQ, TL, MO. But the ratio of the circles to the circles is equal to the ratio of the cylinders which stand on them to one another, if the heights of the cylinders are equal. So the cylinders whose bases are the circles with radii FC, TK, MN and whose heights are the lines EF, FT, TM are less than two thirds of the cylinders whose bases are the circles with radii EB, FQ, TL, MO and whose heights are the equal lines EF, FT, TM, MA. But the cylinders whose bases are the circles with radii lines FC, TK, MN and whose heights are the lines EF, FT, TM are the sawed-off solid whose basis is the circle with radius RE* and whose top is the circle with radius MN, which is in the interior of the hemisphere. And the cylinders whose bases are the circles with radii the lines EB, FQ, TL, MO and whose heights are the lines EF, FT, TM, MA are the cylinder BH. So the sawed-off solid in the interior of the sphere is less than two thirds of the cylinder BH.

But it had been proved that this sawed-off solid is greater than two thirds of cylinder BH, and this is impossible. [319] But this impossibility is a consequence of our assumption that the hemisphere is greater than two thirds of the cylinder BH. So the hemisphere is not greater than two thirds of the cylinder BH.

I say: the hemispere is also not less than two thirds of the cylinder BH.

If this were possible, then let it be less than two thirds of cylinder BH, and let the difference between the hemisphere and two thirds of the cylinder be the magnitude t. Then the magnitude t is less than the cylinder BH.

If half of the cylinder BH is cut off, and half of the remainder, and half of the remainder, in the way which we have explained, then (eventually) there will be a remainder less than the magnitude t. The part of the cylinder which remains after the division of it in the way which we have explained, consists of the round solids which are produced by the rotation of the rectangles BC, CK, KN, NA and corresponding rectangles, such that the surface of the sphere passes through their interiors. Let the division end with something (i.e., a remainder) less than the magnitude t, and let this (i.e., the remainder) consist of the round solids produced by the rotation of the rectangles BC, CK, KN, NA. Then the parts of those round solids outside the sphere are much less than the magnitude t. But the hemisphere together with the magnitude t was (assumed to be) equal to two thirds of the cylinder BH. So the hemisphere together with the parts of the round solids outside the hemisphere are much less than two thirds of the cylinder BH. But the hemisphere together with the parts of the round solids outside it are the sawed-off solid whose base is the circle with radius EB and whose top is the circle with radius AW, which (solid) circumscribes the hemisphere. So the sawed-off solid is less than two thirds of the cylinder BH.

It has been shown before that the squares of the lines FC, TK, MN are less than two thirds of the squares of the lines EB, FQ, TL, MO, and that the difference is less than two thirds the square of EA. So if we add to the squares of the lines FC, TK, MN the whole square of EB, which is equal to the square of EA, the sum of the squares of the lines EB, FC, TK, MN is greater than two thirds of the squares of EB, FQ, TL, MO. So the circles whose radii are the lines EB, FC, TK, MN are greater than two thirds of the circles whose radii are the lines EB, [321] FQ, TL, MO. And the cylinders whose bases are the circles with radii lines EB, FC, TK, MN and whose heights are the lines EF, FT, TM, MA, which are equal to one another, are greater than two thirds of the cylinders whose bases are the circles with radii the lines EB, FQ, TL, MO and whose heights are the lines EF, FT, TM, MA. But the cylinders whose bases are the circles with radii the lines EB, FC, TK, MN and whose heights are the lines EF, FT, TM, MA are (together) the sawed-off solid whose base is the circle with radius EB and whose top is the circle with radius AW, that is the sawed-off solid which circumscribes the sphere. And the cylinders with bases the circles with radii the lines EB, FQ, TL, MO and heights the lines EF, FT, TM, MA are (together) the cylinder BH.

So the sawed-off solid which circumscribes the sphere is greater than two thirds of the cylinder BH. But it had been shown that this sawed-off solid is less than two thirds of the cylinder BH. This is impossible.

This impossibility is a consequence of our assumption that the hemisphere is less than two thirds of the cylinder BH. So the hemisphere is not less than two thirds of the cylinder BH. But it had also been shown that it is not greater than two thirds of the cylinder BH. So since the hemisphere is not greater than two thirds of the cylinder BH and also not less than two thirds of it, it is two thirds of the cylinder BH. The whole sphere is twice the hemisphere, and the cylinder whose base is the circle with radius line BE and whose height is line AG, which is the diameter of the sphere, that is twice line AE, is twice the cylinder [323] BH. So the sphere $ABGD$ is two thirds of the cylinder whose basis is the greatest circle in the sphere, and whose height is equal to the diameter of the sphere. That is what we wanted to prove.

End of the treatise on the volume of the sphere.

Source: R. Rashed, ed. Les Mathématiques infinitesimales, *vol. 2, London: al-Furqān Islamic Heritage Foundation, 1993, pp. 307-323. Some errors in the edition have been corrected; the passages have been indicated by asterisks * and pointed brackets $< >$. Explanatory additions made by Jan H. appear in squarebrackets [ ]*