1 Preface

It is unfortunate that one of the most important mathematics books of the medieval Islamic civilization [1] has not been fully translated to English before. In fact, until the middle of the twentieth century it was unknown to modern researchers [8]. According to Rashed [32], traditional history of mathematics was shaken by Luckey's discovery. *Miftah al-Hisab* is written in Arabic and it has never been fully translated to another language except for Russian [9]. Also, two small sections of *Miftah* are translated to English, one on root extraction [5], another one on measuring the areas of muqarnas [33]. Before *Miftah* was discovered by modern researchers, the discovery of decimal fractions was incorrectly attributed to Simon Stevin [10]. *Miftah* contains a systematic treatment of decimal fractions, yet al-Kashi is not the inventor of decimal fractions. According to best available sources, decimal fractions were first introduced in the middle of the tenth century by Abu al-Hasan Ahmad ibn Ibrahim al-Uqlidisi [28]. It is astonishing that false information about the invention of the decimal fractions is still repeated in books published in the 21st century (e.g. [11]) and when they discuss the history of decimal fractions they leave out the most relevant figures such as al-Uqlidisi and al-Kashi. Yet another indication of the fact that the translation of Miftah is way overdue.

The story of Miftah and the discovery of the contributions of al-Kashi to mathematics and astronomy by modern researchers is a reflection of a larger story in the history of science that is called "the classical narrative" by George Saliba [12]. One of the tenets of this narrative is the assumption that the scientific progress and production in the Islamic World stopped well before the 15th century. This Euro-centric version of history of science was

[1] We will be using the terms "Islamic Civilization" and "medieval Islamic Civilization" in a very broad sense. We are particularly referring to the medieval Islamic civilization for which the time period extends approximately from late 7th century to the 16th century (inclusive). Geographically, it spans a large region –from Spain to the west to China, and India to the east. Therefore, it encompasses much diversity in terms of languages, ethnicity, and cultures. It also contains many different political powers and organizations (such as Umayyads, Abbasids, Fatimids, al-Andalus, Seljuks, Ottomans, Safavids, and others). It was ethnically and religiously highly diverse in which individuals from many different backgrounds contributed to scientific knowledge and progress, being a Muslim was not a prerequisite for this contribution. Arabic was the language of science during this time period, and it has been the language of religious studies. Once again, we use the term in a very broad sense and in no way do we imply a monolithic culture or civilization.

written with a huge deficit in research on primary sources of scientific works in the Islamic civilization and it has been around for a long time. Since the second half of the 20th century much research has been conducted to remedy this situation and a lot of progress has been made. However, more work is needed on two fronts. i) There is still a wealth of primary sources waiting to be studied in many parts of the world, and ii) more efforts is needed to disseminate accurate information about Islamic science to counter and repair the damaging effects of the classical narrative.

The personal story of the first author of this book is also really instructive to illustrate the damaging effects of the classical narrative. Being born and raised in the capital of the Ottoman Empire (that is Istanbul in modern day Turkey), I have been unaware of the profound contributions of Islamic scholars to mathematics and science until a few years ago. It was disappointing to learn that the unique copy of al-Uqlidisi's *Kitab al-fusul fi al-Hisab al-Hindi* was in Yeni Cami Library in my hometown, yet nobody studied it until 1960's [27]. When it was finally translated to English [28], incorrect attributions about the origin of decimal fractions had been circulating for a long time. I found manuscript copies of Miftah in Süleymaniye library (again in my hometown) and I realized that they seem to be unknown in the literature.

The impacts of the classical narrative are evident both in the west and the Islamic world. Today, many people in the Islamic countries are either totally unaware of great Islamic scholars such as al-Uqlidisi,ibn al-Haytham, ibn Shatir, al-Tusi, al-Kashi ... or they vaguely heard some things about their contributions but it is mixed with misinformation (e.g. Muslims invented zero) or full of exaggerations. Therefore, studying primary sources, making them available to a larger audience, and disseminating research findings to the general public are very important activities. We hope that publishing *Miftah al-Hisab* will serve this purpose well.

We are thankful to ....

2 A Biography of al-Kashi and a Brief History

On March 2, 1427, Ghiyath al-Din Jamshid bin Masud bin Mahmood al-Tabeeb Al-Kashi (or al-Kashani) completed a monumental book in Arabic on arithmetic called *Miftāh al-hisāb*. The most commonly used translation of this title, the one which we prefer as well, is "Key to Arithmetic". Other possible translation include "Calculator's Key" and "Key to Calculation". Henceforth we will be referring to it as *Miftah* and its author as al-Kashi. This encyclopedic work includes three general subjects: arithmetic, geometry, and algebra. It comprises five treatises on arithmetic of integers, arithmetic of fractions, arithmetic of sexagesimal numbers, geometry and measurement, and algebra. Al-Kashi dedicated the *Miftah* to Ulugh Beg who himself was a prominent scientist and was the ruler of Samarkand at the time. In Berggren's assessment *Miftah* was "the crowning achievement of Islamic arithmetic, and truly a gift fit for a king" ([4], p. 22).

We do not know all the details of al-Kashi's life. He was born in Kashan, in modern day Iran some 150 miles south of the capital city Tehran, in the later part of the 14th century but we do not know for certain in what year. Some authors seem to think that he was born in 1380 but the source of this date is not clear. Therefore, he is also (perhaps more accurately) known as al-Kashani but al-Kashi is more commonly found in the literature. According to Suter, al-Kashi died in the year 1436 [13]. However, Kennedy gives the date June 22, 1429 (Ramadan 19, 832 A.H.) based on a note on a manuscript copy of one of al-Kashi's works [14] which is the commonly accepted date of al-Kashi's death.

His given name is Jamshid, his father's name is Masud and his grandfather's name is Mahmood. He has several nicknames best known of which is al-Kashi. Giyath al-Din means the rescuer of the religion. Al-Tabeeb means physician. This is due to the fact that he practiced medicine as a profession before devoting his time totally to the study of mathematics and astronomy later in his life. He is known as al-Kashi (or al-Kashani) because he is from Kashan. Assuming a nickname based on the place of birth was common in the Arabic and Islamic culture. There are other famous Islamic scholars who are best known by such nicknames such as al-Khwarizmi and al-Tusi.

The first dated event that we know of about his life is the observation of a lunar eclipse in his hometown of Kashan on June 2, 1406, a date recorded

Figure 1: Iranian stamp featuring an image of al-Kashi with an astralobe in the background

in his *Khaqani zij* as the first date of a series of lunar eclipses he observed. During al-Kashi's early years, Timur (Tamerlaine, 1336-1405) was conquering vast regions including central Iran. It was a difficult time for the people of the region with widespread poverty and turmoil. Apparently, al-Kashi managed to study mathematics and astronomy during this time. The conditions improved after Shah Rukh Mirza (1377-1447) took over the reign upon his father's death. Al-Kashi's best times came after the young prince Ulugh Beg (1394-1449) became the governor of Samarkand, a city in Transoxiana in modern day Uzbekistan, after Shah Rukh Mirza moved the capital to Herat in modern day Afghanistan. Prince Ulugh Beg was a scientist himself excelling in mathematics and astronomy. Additionally, he supported many scholars and students. He gathered some of the best scholars of his time in Samarkand, and established an observatory there which was one of the best observatories ever built until that time. Ulugh Beg also established a madrasa –a school for advance study in theology and sciences– in Samarkand between 1417 and 1420. The construction of observatory began after the completion of the madrasa [22]. Therefore, Samarkand became a major center of research and learning of the time.

We know that al-Kashi joined the scientific circle of Ulugh Beg upon his invitation. We do not know exactly when al-Kashi moved to Samarkand

but it must be around 1420. Different dates are mentioned for this move by researchers such as 1417 in ([19], p. 6), 1418 in ([4], p. 21), and 1421 in ([36]). By the time he was invited to Samarkand, al-Kashi must have proved his scientific ability. Indeed, we know that he completed several works between 1406 and his move to Samarkand. For example, in 1407 he completed *Sullam al-sama -Ladder of the Heaven on resolution of difficulties met by predecessors in determination of distances and sizes*. In 1410-1411, he wrote *Mukhtasar dar ilm-i hay'at-Compendium of the science of astronomy* dedicated to Sultan Iskandar, one of the rulers of Timurid dynasty. In 1413-1414, al-Kashi wrote *Khaqani zij* and dedicated it to Ulugh Beg. In the introduction of this book al-Kashi complains about living in poverty while working on mathematics and astronomy, and he says he could not have completed this work without Ulugh Beg's support. Al-Kashi sought to get patronage of a ruler and this might be his first success.

Al-Kashi did some of his best work in Samarkand under the patronage of Ulugh Beg. This includes *Miftah al-Hisab*, an encyclopedic book on elementary mathematics, and his remarkable approximations to π and $\sin(1^\circ)$. Two letters of al-Kashi in Persian to his father give us interesting facts and insights into the scientific environment of Ulugh Beg's court and personalities of some of the important scholars there. From these letters and other sources we learn that Ulugh Beg tolerated al-Kashi's lack of court etiquette thanks to his excellent command of mathematics and astronomy.

The letters of al-Kashi were discovered in the second half of the 20th century but at different times. The second letter was discovered first and published in 1960 by two different researchers independently. Kennedy gave an English translation of the letter together with a commentary in [34], and Sayili gave both a Turkish translation and an English translation in [35]. The first letter was discovered later by Bagheri who published it with English translation [36]. Al-Kashi explains that he wrote the second letter and repeated a lot of information in case the first letter, which was sent via the merchants of Qum, might have been lost. From the details given in the letters we can infer that al-Kashi's father was a learned man who knew mathematics and astronomy. The main topics he discusses in his letters are his professional triumphs and how he distinguished himself from other scientists in the court, character of his patron Ulugh Beg, and the progress in the construction of the observatory that was being built when the letters were written. In describing his scientific achievements, al-Kashi speaks of most of the scientific staff of Ulugh Beg in disdain and thinks how incompetent they are. He has a very

high opinion of Ulugh Beg himself, and only very few of other scientists in the group such as Qadizade al-Rumi (1364-1436) who was a teacher of Ulugh Beg. The letters picture Ulugh Beg more as a scientist and scholar than as a ruler or statesman. He attended many of the meetings in which scientific matters are discussed by the scholars of his court. We learn that Ulugh Beg visited Maraga observatory in his childhood which probably made an impression on him. We also learn that serious scientific studies at Samarkand had been undertaken for about twelve years, and that there were at least sixty to seventy mathematicians among Ulugh Beg's staff in addition to astronomers.

Not satisfied with zij's compiled before his time, Ulugh Beg decided to conduct fresh observations and built the Samarkand observatory. It would be reasonable to infer that building of the observatory was planned before al-Kashi arrived at Samarkand and started shortly after his arrival. We learn from his letters that Ulugh Beg followed al-Kashi's advice on certain technical aspects of the building during the construction and changed his original plan upon being convinced by al-Kashi's explanations. Al-Kashi was the first director of the astronomical observations at the observatory [35] but he passed away before the study was complete and Qadizade al-Rumi took over the project. His life was not long enough either to complete the project. Ali Qushji (1403-1474) completed the project. Al-Kashi's letters are among the best sources for information on the instruments of the Samarkand observatory.

Al-Kashi praises Ulugh Beg in these letters in several ways. For example, he is well versed both in religious sciences and mathematics. (Translation from [35])

> Truth is that, first of all, he knows most of the holy Quran by heart, and he has a ready knowledge of its exegeses. For each occasion he cites an appropriate verse of the Quran, and he makes elegant quotations. Every day he reads fluently and in the proper manner two sections from the sacred book in the presence of experts who know the whole of the Quran by memory, and no mistakes occur. His knowledge of grammar and syntax is very good, and he writes Arabic extremely well. Likewise, he is well versed in jurisprudence, and he is acquainted with logic and the theory of literary style, as well as with the principles of prosody.
>
> His majesty has great skill in the branches of mathematics. His accomplishment in these matters reached such a degree that one day,

> while riding, he wished to find out to what day of the solar year a certain date would correspond which was known to be a Monday of the month of Rajab in the year 818 and falling between the 10th and the 15th of the month. On the basis of these data he derived the longitude of the sun to a fraction of 2 minutes by mental calculation while riding on horseback, and when he got down he asked this servant to check his result.
>
> It is true that, as in mental calculation it is necessary to retain quantities in one's mind and to derive others from them, and because in the faculty of memory there is a shortcoming, he [i.e. al-Kashi himself] could not find the result (correctly) in degrees and minutes and was content with degrees only. But it is not given to any person of our time to do the like; no one else is capable of it.

He also says Ulugh Beg is very generous and supports a large number of students seeking knowledge. (Translation from [36])

> His royal majesty [Ulugh Beg] had donated a charitable gift amounting to 30,000 kopaki dinars, of which 10,000 had been ordered to be given to students. [The names of the recipients] were written down: [thus] 10,000-odd students steadily engaged in learning and teaching, and qualifying for a financial aid, were listed.

These numbers show that Samarkand was a major center of learning at the time. This is supported by other facts mentioned in al-Kashi's letters. He says in addition to the students studying with financial aid, there are about 500 persons among notables and their sons who began studying mathematics which was taught at twelve places. He says "there are 24 calculators some of whom are also astronomers and some began [studying] Euclid ['s *Elements*]" [36]. He compares the situation with Kashan where one or two persons may be associated with a given discipline, therefore it is not a good environment for healthy discussion of scientific matters.

Ulugh Beg is also described, in these letters, as a kind and open-minded person who is keen on rigorous investigation of scientific matters. He listens to all points of views, lets everyone express their opinions and make their case, gets engaged in arguments with students and experts. He lets discussions go on until the issues become clear to everyone. He does not approve submission to the authority without convincing proofs [35].

Figure 2: Ulugh Beg's Statue in Samarkand, Uzbekistan. (Image source: wikipedia)

> He is indeed good-natured to the utmost degree of kindness and charity, so that, at times, there goes on, at the madrasa, between His Majesty and the students of the seeker of knowledge so much arguing back and forth on problems pertaining to any of the sciences that it would be difficult to describe it. He has ordered, in fact, that this should be the procedure, and he has allowed that in scientific questions there should be no agreeing until the matter is thoroughly understood and that people should not pretend to understand in order to be pleasing. Occasionally, when someone assented to His Majesty's view out of submission to his authority, His Majesty reprimanded him by saying "you are imputing ignorance on me." He also poses a false question, so that if anyone accepts it out of politeness he will reintroduce the matter and put the man to shame.

A large part of the letters is devoted to communicate to his father that al-Kashi is the best, most proficient, and competent scientist in the group. He makes this point in several ways and gives many examples. He narrates several occasions where other scientists spent a lot of time on a given problem but they could not solve it. When al-Kashi enters the picture he solves the problems quickly and easily. In some cases, other scholars think that there is a missing information in the problem statement but al-Kashi shows that this is not the case and solves the problem. Other times, he publicly challenges other scholars on scientific matters and proves his points. He says this made him famous. He also writes about strong words of praise for al-Kashi by Ulugh Beg about his character and scientific competency [35].

> As to the complimentary remarks of His Majesty, of which mention has been made above, the situation is that no week passes without some friends reporting to this servant that His Majesty made such and such remarks tonight or today concerning me. They are as follows: "He has the knowledge of things ready at hand"; "He knows extremely well"; "His knowledge is superior to that of others"; "His knowledge is more readily at his disposal and more substantial than is the case with Qadizada"; 'His mind works better in this science than that of Qadizada"; "Mawlana Giyath al-Din knows all the parts of this science, and he solves at once or in a single day a difficulty which takes Qadizada ten days to disentangle."
>
> He said, likewise, "He is a good and kind-hearted man. All those

> who have access to our circle, whether they be of the notables or not, have not restrained themselves but have quarreled with people and have transgressed their limits although I have shown them little courtesy. On the other hand, although I have extended much courtesy and many grants to Mawlana Giyath al-Din and even though he is always honored with access to our company, he has not had any quarrels with anybody and nor has anyone complained about him." "He has not, out of greed, resorted to speak and gossip behind people's backs"; "He conducts himself very well." He has said things of this nature many times.

Al-Kashi thinks that Qadizada is the most knowledgeable scholar in the group but he narrates situations that show he is not as good as Al-Kashi. An immediate logical consequence of this is that al-Kashi is the very best scholar in the entire group of Ulug Beg's scientific staff. He tells his father that he followed his advice to focus on only one subject (in this case it is astronomy) and he understands the reason behind this advice as "because occupation with another subject would indeed distract me from astronomical observations; second, because of my occupation with another art in which I may be a beginner, there may occur in my discussions or compositions some defect or error which people would bring to bear on the other arts [in which I am adept]" [36]. Evidently, another purpose of al-Kashi's letters to his father was to refute some rumors his father heard through a person named Shams al-Din. He makes it clear that Shams al-Din's statements about him are false.

Al-Kashi had a productive time in Samarkand but unfortunately it did not last long. He died on June 22, 1429 at the observatory in which he was deeply involved. Ulugh Beg says the following for al-Kashi in the preface of his own zij written some years after al-Kashi's death: "the remarkable scientist, one of the most famous in the world, who had a perfect command of the sciences of the ancients, who contributed to its development, and who could solve the most difficult problems" [14].

Figure 3: Ulugh Beg's Observatory in Samarkand, Uzbekistan. (Image from Wikimedia Commons)

3 List of Al-Kashi's Known Works

In addition to *Miftah* which is the most comprehensive work of al-Kashi, we know of the following works of al-Kashi. Al-Kashi's works were collected in *Majmu'*(*Collection*), Tehran 1888. The following list of al-Kashi's works can be found in [1] and [14].

- **Zij Khaqani**
  Zij is a Persian word used for astronomy books containing tables that are important for astronomical calculations. This work was written around 143-1414 before al-Kashi moved to Samarkand. Written in Persian, it was an update and completion of the *Zij Ilkhani* compiled by famous Nasir al-Din al-Tusi about 150 years earlier. Bartold believed that al-Kashi dedicated this work to Shah Rukh who was a patron of science in Herat [15]. However, Kennedy established that the zij was dedicated to Ulugh Beg, son of Shah Rukh and ruler of Samarkand [14]. This may be al-Kashi's first attempt to secure funding under Ulugh Beg's patronage. Al-Kashi used iterative methods to obtain approximate value of the third of each arc [1]. This zij contains six treatises and the tables in it are in sexagesimal system [14]. Al-Kashi also states that he collected data from the works of earlier astronomers (astrologers) that did not show up in other tables, along with geometric proofs [1]. Arabic manuscripts are available in London, Oxford, and Istanbul [14].

- **The Treatise on Circumference**
  One of al-Kashi's most remarkable achievements is his approximation of π accurate to 9 sexagesimal places, or 16 decimal places. According to Hogendijk [6], this is one of the highlights of the mediveal Islamic mathematical tradition. Before al-Kashi, the previous record was 6 decimal place approximation by Chinese mathematicians. It took nearly two centuries before the Dutch mathematician Ludolf Van Ceulen surpassed al-Kashi's accuracy with 20 decimal places. Written in 1424, the original title of this work is *al-risala al-muhitiyya*. His motivation to compute π so accurately was to calculate the circumference of the universe so that the round-off error in the approximation would be no more than the width of a horse's hair. He set the accuracy of his approximation in advance and he used a polygon with $3 \cdot 2^{28} = 805,306,368$ sides. Hogendijk states that since al-Kashi's text was not available in

English translation until recently, incorrect or confused statements on the history of π often appeared in the western literature [6]. For example, Al-Kashi was not mentioned in *A History of Pi* [6], a popular book on the subject. Arabic manuscripts are available in Istanbul, Tehran and Meshed [14].

- **The Treatise on Chord and Sine**
  Given al-Kashi's dissatisfaction with earlier zijs, he calculated accurate values for sines and chords in this work to help with astronomical tables. He also calculated $\sin(1^\circ)$ to the same degree of accuracy of his computation of π. Unfortunately, the original text of this work is lost, however, Qadizadeh al-Rumi wrote an account of this treatise in 1299 which is available in the national library of Iran [24]. It was translated to Russian. The outline of al-Kashi's main method in this approximation is as follows. He first used the trig identity $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$ which he transformed to a cubic equation of the form $x = a + bx^3$. He then used an iterative method to obtain successive approximations to $\sin(1^\circ)$ that get more accurate at each step. His beautiful approach is still important in modern mathematics and is usually covered under the name of "fixed point iteration" in numerical analysis courses. See [37] for more details on the computation of $\sin(1^\circ)$. There is an edition of this manuscript in *Majmu'* [14].

- **The Zij at-Tashilat- The Zij of Simplifications**
  Al-Kashi refers to this work in *Miftah* among his works but it is non-extant. It probably included a simplified method of computing the positions of celestial bodies [22].

- **Risala dar Sharh-i Alat-i Rasd– Treatise on the Explanation of Observational Instruments**
  Written for Sultan Iskandar (Kara Yusuf of Black Sheep Turks (Karakoyunlu) dynasty) in 1416, al-Kashi gives brief yet accurate descriptions for the constructions of eight astronomical instruments. A manuscript copy of this is available in Leiden, which, according to F. F. Bartold, was written by al-Kashi himself [1]. Persian manuscripts are available in Leiden and Tehran[14].

- **Nuzha al-Hadaiq–Delight of Gardens**
  Al-Kashi wrote this treatise in 1416 (February 10, 1416) in Kashan to

which he made additions in Samarkand in 1426. A revised Persian version was written by an anonymous astronomer in Istanbul around 1490. [7]. It gives an explanation of how to build an instrument invented by al-Kashi which he called "The Plate of Conjunctions" [1]. According to al-Kashi, it is an instrument from which one can retrieve the calendars of planets, their widths, dimensions, distances from earth, eclipses and other related matters. It is possible to describe this instrument, which has a similar shape as the astrolabe, is the diagram for the approximate graphical solutions of many matters related to the movement of the stars based on the mean values of their coordinates. His additions include ten appendices which describe additional techniques to utilize the instrument [22]. Arabic manuscripts are available in London, Dublin and Bombay [14].

- **Sullam al-Sama –The Ladder of the Sky or The Stairway of Heaven**
  Al-Kashi completed this work in his hometown in 1407. It is the earliest known work of al-Kashi.The full title of this text in astronomy includes "on Resolution of Difficulties Met by Predecessors in Determination of Distances and Sizes (of Heavenly Bodies)" [10]. Al-Kashi mentions this treatise in the preface of *Miftah*. He implies that scholars before him had difficulties and disagreements about distances and sizes of the heavenly bodies. So, he decided to write this book to help future scholars. Arabic manuscripts are available in London, Oxford, and Istanbul [14]. There is an Iranian TV series with the same title, the Ladder of the Sky, about al-Kashi's life which was broadcast during the month of Ramadan of 2009. Full episodes are available at http://www.shiasource.com/ladder-of-the-sky

- **Talkith al-Miftah–Abridged Key [to Arithmetic]**
  Written before the *Miftah* itself (in 1421), it was an early and abbreviated version which contains three treatises and about one eighth of the material in *Miftah* [24]. Qurbani provided Persian translations of the chapters of *Talkhith* [25]. Arabic manuscripts are available in London, Tashkent, Istanbul, Baghdad, Mosul, Tehran, Tabriz, and Patna [14].

- **Miftah Al Asbab-fi al-Ilm al-Zij– The Key of Reason in the Science of Astronomical Tables**
  There is an Arabic manuscript in Mosul [14].

- **Risala dar Sakht-i Asturlab–The Treatise on the Construction of an Astrolabe**
  There is a Persian manuscript in Mosul [14].

- **Ta'rib al-Zij–Arabization of the Zij**
  This is an Arabic translation of the introduction of Ulugh Beg's zij which was in Persian. Manuscripts are available in Leiden and Tashkent [14].

- **Ilkahat an-Nuzha–Supplement to the Excursion, 1427**
  There is an edition of a manuscript in *Majmu'* [14].

- **Wujuh al-Amal al-Darb fi'l-Takht wa'l-Turab– Ways of Multiplying Using a Dust Board**
  There is an edition of an Arabic manuscript in *Majmu'* [14].

- **Nataij al-Haqaiq– Results of Verities**
  There is an edition of an Arabic manuscript in *Majmu'* [14].

- **Risala fi Ma'rifa Samt al-Qibla min Daira Hindiyya Ma'rufa–Treatise on the Determination of Direction of the Qibla from a Circle Known as Indian Circle**
  There is an edition of an Arabic manuscript in Meshed [14].

4 Manuscript Copies of *Miftah*

Two different typeset and printed books of *Miftah* that include the original Arabic manuscript together with some commentary and explanations are available. The first one is by Amad Sa'id al-Dimirdash and Muammad Hamdi al-Hifni al-Shaykh published in Cairo in 1969 [2] in which commentaries are in Arabic. The other one was published in 1977 in Damascus and the author is Nabulsi [1]. Nabulsi edition has most commentaries in Arabic and some in French as well. It lists the following known manuscript copies of *Miftah* at the time of its publication.

1. Orginal book of Miftah al-Hisab, written by Jamshid bin Mas'ud bin Mahmoud al-Kashi March 3rd, 1427 (830 h.), but it is missing.
2. Al Burgandy's manuscript, which is also missing. According to Muhammad Al-Sadiq Al-Arassengi in the Zahiri manuscript, it is written by Abdul Ali Al-Burgandy on Tuesday, 17th day of Dhu al-Hijja in 889 H., which corresponds to January 5, 1485.
3. The Leiden manuscript, written in 1558 (965 h).
4. The British Museum manuscript, written in 1589 (887 h).
5. The Zahiri manuscript written in 1691 (1102 h).
6. The Leningrad manuscript, written in 1789 (1204 h).
7. The Scientific Library of Prussia manuscript in Berlin, written in 1886 (1303-1304h).
8. The Public Scientific Library in Berlin manuscript (spr 1824)
9. The Berlin Institute for History of Medicine and Science, number 1 and 2.
10. The National Library of Paris, number 5020.

Dimirdash and Muhammad Hamdi edition also mentions the manuscripts 3,4,6-10 and they say that they relied on Leiden manuscript [2]. Their list does not include the Zahiri manuscript but includes one that does not appear in the list above, a stone-print copy in Tehran, located in the Tayomrian

cabinet number 255-Math. Nabulsi says he used the following manuscripts: Zahiri, London, and Leiden as well as the print book [2] and the Russian translation by Rosenfeld and Youschkevitch published 1956 [9], and the Zahiri manuscript is the main source his investigation and original Arabic text. Nabulsi regards Zahiri manuscript as the oldest known one, gives the following information about it ([1], p. 31) and includes a picture of the first and last pages.

> The Zahiri manuscript, number 7795 amongst the collection of books in the Zahiri library in Damascus. It is written on 128 sheets of paper of dimensions 21x12 cm, the handwriting is nice, the titles are written in red, has commentaries, it has eroded from the sides, embroidered with gold (written by Abd Al Ali bin Muhammad Al Burgandi in 889 then copied by Ibn Muhammad Jafar Sadiq Al Arasinji in 1103).

Nabulsi notices that time intervals between the original work and its subsequent manuscripts listed above are 73 years, 31 years, 102 years, 98 years and 97 years. He concludes that there must be several other manuscripts of *Miftah* that are still missing to this day. The manuscript we are using is not in Nabulsi's list, and it appears to be one of the manuscripts he suspected was missing. This manuscript is in Süleymaniye library in Istanbul in its Nuruosmaniye branch with new record number 2532 and old record number 2967. It was written in Ramadan 854 H., which corresponds to October 1450. There are several other manuscripts of *Miftah* in Süleymaniye library which are labeled as Atıfefendi1719, Esadefendi13175, Fatih5421, Hamidiye883, and Hüsnüpaşa1268. Some of them appear to be incomplete and some of them do not seem to contain a date of completion. The manuscript that we are using, Nuruosmaniye2967, is the oldest one in the collection, and it is complete. Based on available data in the literature, Nuruosmaniye2967 manuscript seems to be one of the oldest, possibly the oldest, available manuscript copy of *Miftah* to this date. Note that the Zahiri manuscript that Nabulsi considers "the oldest known manuscript of *Miftah* so far" was originally written in 889 H. (1484 AD) which is later than the date of Nuruosmaniye manuscript we use (854H-1450 AD).

5 Modern Researchers' Assessment of Miftah and al-Kashi's Work

We include here a few selected quotes from modern researchers about the significance of al-Kashi's work in general, and *Miftah al-Hisab* in particular.

- Key to Arithmetic is the crowning achievement of Islamic arithmetic (J. J. Berggren, [4], p. 22)
- It was only 1948 that Luckey presented the first extensive and detailed study of al-Kashi's work, and the traditional historical picture was explicitly shaken. (R. Rashed, [32], p. 149).
- In the richness of its contents and in the application of arithmetical and algebraic methods to the solution of various problems, including several geometric ones, and in the clarity and elegance of exposition, this voluminous textbook [that is, *Miftah*] is one of the best in the whole of medieval literature; it attests to both the author's erudition and his pedagogical ability. Because of its high quality the Miftah was often recopied and served as a manual for hundreds of years. (A. P. Youschkevitch and B. A. Rosenfeld, [14], p. 256)
- It is in the work of Giyath al-Din Jamshid al-Kashi in the early fifteenth century that we first see both a total command of the idea of decimal fractions and a convenient notation for them ... (V. J. Katz, [21], p. 270)
- The power and elegance of computational algorisks developed by Islamic mathematicians of the ninth through the fifteenth centuries has only recently come to be appreciated. The most sucessful of these computers was the Iranian scientist, Jamshid Ghiyath al-Din al-Kashi (E. S. Kennedy, [23], p. 522)
- Al-Kashi was first and foremost a master computer of extraordinary ability, witness his facile use of pure sexagesimals, his wide application of iterative algorisms, and his sure touch in so laying out a computation that he controlled the maximum error and maintained a check at all stages. (E. S. Kennedy, [22], p. 8)
- In the determination of π, and in computational mathematics as a whole, al-Kashi was a pioneer. (J. P. Hogendijk, [6], p. 85)

- Key to Arithmetic was used for centuries by astronomers, architects, artisans, surveyors, and merchants as a textbook in the Islamic world (J. Freely, [17])

- Its [*Miftah*'s] influence even extended to reach the mathematics teaching climate in Europe (F. Riahi [18])

- Al-Kashi's most impressive mathematical work was *Key to Arithmetic*. The work is a major text intended to be used in teaching students in Samarkand, in particular al-Kashi tries to give the necessary mathematics for those studying astronomy, surveying, architecture, accounting and trading. (J. J. O'Connor and E. F. Robertson, [10])

- The observatory, as the scientific institution we know today, was born and developed in the Islamic world (J. L. Berggre, [4], p. 21))

- From the standpoint of longevity and work therefore the Samarqand Observatory was one of the most important observatories of Islam, and it probably was the most important... Giyath al-Din Jamshid and Muin al-Din-i Kashi prepared the plan for the observatory... Giyath al-Din was the first director of the Samarqand observatory ... (A. Sayili, [26], p. 265, 266, 271.)

- There is little doubt that al-Kashi was the leading astronomer and mathematician at Samarkand and he was called the second Ptolemy by an historian writing later in the same century. (J. J. O'Connor and E. F. Robertson, [10])

- Fulfilling the needs of accountants, engineers, mathematicians, surveyors, lawyers, and others, this book is unprecedented amongst mathematical works in the middle ages in terms of its perfection, organization, and clarity of explanation (N. Nabulsi, [1], p. 28)

- It [root extraction algorithm of al-Kashi in *Miftah*] is an example of the powerful and sophisticated numerical methods developed in the Muslim East, methods yielding results of unprecedented precision for their time, and which reached their culmination in the work of al-Kashi(A. Dakhel, [5], preface, v viii)

- The achievements of al-Kashi represent the culmination of Islamic civilization's progress in mathematics. Some of these achievements reappeared or rediscovered later in Europe. (F. Sezgin, [31], p. 66)

- Not only was this [approximation of $\sin(1^\circ)$] the most fascinating and creative method of approximation, but it was the first approximation method in the history of mathematics, and the most significant achivement in medieval algebra (M. K. Azarian, [24], p. 39)

- In the Ottoman Empire mathematicians called any excellent mathematician by the name al-Kashi (O. H. Taani, [19], p. 4 quoting from [30] p. 236)

- My dissertation demonstrates that there is a great need to have a complete translation of *Miftah* to English, and a more detailed description of al-Kashi's mathematics and methods. (O. H. Taani, [19], p. 4)

6 Pedagogical Aspects of Miftah

It is clear that *Miftah* was written as a textbook and a practical guide for people in various professions. It is clearly not a theoretical book as it contains no proofs or justifications for algorithms or procedures. The title suggests that arithmetic is viewed as the key in solving any problem that requires calculations. Hence al-Kashi presents mathematics needed in such fields as engineering, astronomy, accounting, surveying, arts, architecture, and business. In the introduction of *Miftah*, al-Kashi describes arithmetic as follows: "Arithmetic is the science of rules to determine numerical unknowns from specific known quantities". Al-Kashi also makes his purpose of making it a practical guide for anyone who needs arithmetic and elementary mathematics clear in the introduction:

> I solved many problems I was asked by expert mathematicians either for testing me or for their own learning. Some of those problems could not be solved by one of the six algebraic forms. Through these works I acquired a lot of knowledge to solve elementary mathematical problems in the easiest, most beneficial and most efficient ways using clear methods. I wanted to clarify and compile them so that it becomes a guide to anyone interested. Therefore, I wrote this book and collected all that professional calculators need, avoiding tiring length and annoying brevity. I presented rules for most operations in tables to make them accessible for engineers.

Taani studied the content and pedagogy of *Miftah* [19, 20]. His main findings about al-Kashi's pedagogy in *Miftah* are what he calls "multiple paths to practice mathematics" and his "exhaustive exclusive classification methods". Taani's formulation of "multiple paths" include the following features: multiple definitions, multiple algorithms, multiple formulas, and multiple solutions [20]. He explains that what inspired him to consider investigating multiple paths in al-Kashi's pedagogy was student comments. He initially used some excerpts from al-Kashi in his pre-calculus class and his goal was to "get data from students about using historical sources in the classroom". When he administered a questionnaire about the lesson, student comments on al-Kashi's method of using multiple methods/approaches to solve a given problem captured his attention and that became a major topic in his dissertation.

As an example of multiple definitions of mathematical concepts, consider these two definitions of division by al-Kashi in *Miftah*

> i) In integers, it is dividing of the dividend in units of the divisor into a number of equal parts so that each share from the divisor is fixed. Such a share is called the quotient.
>
> ii) Its general definition is to obtain the number whose ratio to one is the same as the ratio of the dividend to the divisor.

Al-Kashi often presents multiple methods to perform mathematical calculations. For example, he presents five different ways of performing multiplication of integers. He gives three different formulas to compute the area of a generic triangle, and four different formulas specifically for computing the area of an equilateral triangle. The last treatise of *Miftah* contains a number of word problems and problems in inheritance (determining share of each heir according to Islamic inheritance laws). He shows multiple ways of solving these problems.

Taani suggests that his work may be used in a classroom in two different ways [20]. The first is directly using the primary text of al-Kashi will let the students live the experience of discovering mathematics in the 15th century. The second is to follow al-Kashi in presenting certain topics from multiple perspectives. He comments that the benefits of using multiple paths include: *providing flexibility of thinking*, *providing opportunities for comparison*, *creating a network of ideas*, and *improving creativity* [20].

Therefore, in addition to its profound significance in the history of mathematics due to its content, *Miftah al-Hisab* is also worthy of attention for its pedagogical aspects.

7 Possible Future Projects

Not only does Al-Kashi not give proofs or justifications for his algorithms and procedures, but he does not give any information about their origins. It is not clear what parts are his inventions or contributions and what parts are considered established results. A possible future work could be to research the history of the algorithms (such as finding square roots and higher degree roots) described in *Miftah* to determine what exactly the contributions of Al-Kashi might be. On a different direction, it is quite likely that Al-Kashi's work may have influenced mathematicians who came after him. In a recent article [29] we pointed out one such possible connection. We observed that the famous Flemish mathematician Simon Stevin (1548-1620) presented root finding algorithms in his well known book *L'arithmétique"*, ("Arithmetic") published in 1585. On the surface, Stevin's algorithms look much different from Al-Kashi's. However, when we examine them carefully we observe that the underlying algorithm is the same, though there are some curious differences in details and implementation (see [29] for more info). To the best of our knowledge, no direct connection between the two mathematicians is known. Given Al-Kashi's influence on Ottoman mathematicians and the education system [30], one might imagine flow of ideas from Central Asia to Europe through the Ottoman land. This is an area that needs further research.

References

[1] Al-Kashi. *Miftah a-Hisab*. (edition, commentary and notes by Nabulsi Nader). Damascus: University of Damascus Press, 1977.

[2] Al-Kashi. *Miftah a-Hisab*. (edition, and commentary by Ahmad Sa'id al-Dimirdash and Muhammad Hamdi al-Hifni al-Shaykh). Cairo, Dar al-Katib al-Arabi, 1969.

[3] Berggren, J. L. *Episodes in the Mathematics of Medieval Islam*. Berlin, Heidelberg, New York: Springer-Verlag, 1986.

[4] Berggren, J. L. *Episodes in the Mathematics of Medieval Islam*. 2nd, ed, Berlin, Heidelberg, New York: Springer-Verlag, 2016.

[5] Dakhel, A. *The extraction of the n-th root in the sexagesimal notation. A study of chapter 5, treatise 3 of Miftah al-hisab*. W. A. Hijab and E. S. Kennedy, eds, Beirut: American University of Beirut, 1960.

[6] Hogendijk, J. P. "Al-Kashi's determination of π to 16 decimals in an old manuscript", *Zeitschrift fr Geschichte der arabisch-islamischen Wissenschaften* 18, 73-152, 2009.

[7] Hogendijk, J. P., "Sources relating to Ghiyath al-Din Kashani, or al-Kashi", Accesses July 2017, http://www.jphogendijk.nl/arabsci/kashi.html

[8] P Luckey. *Die Rechnenkunst bei Gamsid b. Masud Al-Kasi* (Wiesbaden, 1951).

[9] Rosenfeld B. A., Segal, V. S., Youschkevitch, A. P.*Klyuch Arifmetiki* [Key to Arithmetic], Traktat ob Okruzhnosti, Moscow, 1956.

[10] O'Conner J. J., and Robertson, E. F. Robertson, "The MacTutor History of Mathematics archive". Reterived July 2017 from http://turnbull.mcs.st-and.ac.uk/ history/Biographies/Al-Kashi.html

[11] Devreese, J. T. and Berghe, G. V. *Magic is no magic: the wonderful world of Simon Stevin*, Southampton, UK ; Boston : WIT, 2008.

[12] Saliba, G. *Islamic Science and the Making of the European Renaissance*, The MIT Press, Cambridge MA, 2007.

[13] Suter, H. *Die Mathematiker und Astronomen ....*, pp. 173-174.

[14] Youschkevitch A. P., Rosenfeld, B. A. *Al-Kashi* in Dictionary of Scientific Biography,, vol. 7, New York 1973, pp. 255-262.

[15] Bartold, V. V. *Ulugbek i ego uremya*, p. 108.

[16] Beckmann, P. *A History of Pi*. New Tork: St. Martin's Press, 1971.

[17] Freely, J. *Aladdin's lamp: How Greek science came to Europe through the Islamic World*. New York: Vintage Books, 2009.

[18] Riahi, F. *An early iterative method for determination of* $\sin(1^\circ)$. The College Mathematics Journal 26(1): 16-21, 1995.

[19] Taani, O. H. *An analysis of the contents and pedagogy of al-Kashi's 1427 "Key to Arithmetic" (Miftah al-Hisab)*, PhD dissertation, New Mexico State University, New Mexico State University, ProQuest Dissertations Publishing, 2011. 3534149.

[20] Taani, O. "Multiple Paths to Mathematics Practice in Al-Kashis Key to Arithmetic", *Science & Education* January 2014, Volume 23, Issue 1, pp 125-141.

[21] Katz, V. J. *A History of Mathematics, 3rd ed.* Addison-Wesley, Boston, 2009.

[22] Kennedy, E. S. *The Planetary Equatorium of Jamshid Ghiyath al-Din al-Kashi*, Princeton University Press, New Jersey, 1960.

[23] Kennedy, E. S. *Studies in the Islamic Exact Sciences by E.S.Kennedy, Colleagues and Former Students*, American University of Beirut, Beirut, Lebanon, 1983.

[24] Azarian, M. K. "A summary of the mathematical works of Giyath ud-din Jamshid Kashani", *J. Recreational Mathematics*, 29(1), 32-42, 1998.

[25] Qurbani, A. *Kashani Nameh*, Publication No. 1322 of Tehran University, Tehran, Iran, 1971 (1350 A.H.S.)

[26] Sayili, A. *The observatory in Islam and its place in the general history of observatory* Turkish Historical Society Publication, 1960.

[27] Saidan, A. "The earliest extant arabic arithmetic", *Isis*, 57 (194), 475-490, 1966.

[28] Saidan, A. *The arithmetic of al-Uqlidisi: The story of Hindu-Arabic arithmetic as told in Kitab al-Fusul fi al-Hisab al-Hindi*, translated and annotated by A. S. Saidan, Dordrecht, Reidel Publishing, 1978.

[29] Aydin, N., Hammoudi, L. "Root Extraction by Al-Kashi and Stevin", *Archive for History of Exact Sciences*, 69 (3), 291-310,2015. DOI 10.1007/s00407-015-0150-3

[30] Izgi, C. *Osmanli Medreselerinde Ilim* [Sciences in the Ottoman Schools], Topkapi, Istanbul: Iz Yayincilik, 1997.

[31] Sezgin, F. *Islamda Bilim ve Teknik, Volume I* [Science and Technology is Islam] (translation to Turkish from German by A. Aliy), Turkish Academy of Sciences Publication, Ankara, 2007.

[32] Rashed, R. *The development of Arabic mathematics between arithmetic and algebra*, translated by A. F. W. Armstrong, Dordrecht, Boston, London, Kluwer, 1994.

[33] Dold-Samplonius, Y. "Practical Arabic mathematics: measuring the muqarnas by al-Kashi", *Centaurus* 35(34), 193-242, 1992.

[34] Kennedy, E. S. "A Letter of Jamshid al-Kashi to His Father: Scientific Research and Personalities at a Fifteenth Century Court", *Orientalia* 29, 191-213, 1960; reprinted in Edward S. Kennedy, colleagues and former students, *Studies in the Islamic Exact Sciences*, Beirut: American University, pp. 722-744, 1983.

[35] Sayili, A. *Ulug Bey ve Semerkanddeki Ilim Faaliyeti Hakkinda Giyasuddini Kasi'nin Mektubu: Ghiyath al-Din al-Kashi's Letter on Ulugh Beg and the Scientific Activity in Samarqand*, Ankara: Turk Tarih Kurumu Basimevi, 1960.

[36] Bagheri, M. "A Newly Found Letter of al-Kashi on Scientific Life in Samarkand", *Historia Mathematica* 24, 241-256, 1997.

[37] Aaboe, "A. Al-Kashi's iteration method for the determination of sin(1°)",*Scripta Mathematica* 20 (1954), 24-29, reprinted in IMA56, pp. 354-360.