Problem 7: Euler's Generalization of Fermat's Little Theorem

For a positive integer n, *Euler's phi function* (also called *Euler's totient function*), denoted by $\phi(n)$, is defined as $\phi(n) = \#\{k : 1 \leq k \leq n \text{ and } \gcd(k,n) = 1\}$. i.e., it is the number of positive integers that are less than or equal to n and relatively prime (or co-prime) with n. For example, $\phi(4) = 2$ because among the integers in the set $\{1, 2, 3, 4\}$, two of them (1 and 3) are relatively prime with 4. Euler's Theorem states that if $\gcd(a, n) = 1$ then $a^{\phi(n)} \equiv 1 \mod n$. It is a generalization of Fermat's Little Theorem. Euler's Theorem is in turn a special case of Lagrange's Theorem in group theory.

1. Compute $\phi(1000)$
2. Use Euler's Theorem to compute $2023^{2002} \mod 1000$

As always, show your work, fully explain and justify your answer. A solution that is mainly obtained by a calculator or computer will not be accepted.

Posting Date 3/27/2022. Submit solutions to Noah Aydin, RBH 319 (e-mail or hard-copy, but hard copy submissions must include a time stamp) by noon on Saturday April 9, 2022.