**Solution to Problem of the Week 7**

Starting with a positive real number
$x_0 = a$, let $x_n$ be the sequence of real numbers defined by

$$x_{n+1} = \begin{cases} x_n^2 + 1 & \text{if } n \text{ is even} \\ \sqrt{x_n} - 1 & \text{if } n \text{ is odd} \end{cases}$$

For what values of $a$ will there be terms of the sequence arbitrarily close to 0?

**Solution:** The sequence has terms arbitrarily close to 0 for any positive number $a$. In fact, the subsequence $x_{2n}$ converges to 0. To see this, let $y_n = x_{2n}$. Then we have

$$y_{n+1} = \sqrt{x_{2n+1}} - 1 = \sqrt{x_{2n}^2 + 1} - 1 = \sqrt{y_n^2 + 1} - 1 \qquad *$$

which implies $y_n \geq 0$ for all $n$. It also implies

$$y_{n+1} \leq \sqrt{y_n^2 + 2y_n + 1} - 1 = y_n$$

that is $y_n$ is decreasing. Therefore, (why?) $y_n$ must have a limit, say $L$ where $L \geq 0$. To find the value of $L$, take the limit of each side in * which gives

$$L = \sqrt{L^2 + 1} - 1$$

or

$$(L + 1)^2 = L^2 + 1$$

The only solution to this equation is $L = 0$.