Ben Johnson
Problem of the Week #3 Solution

Observe that the system of equations

$$\begin{cases} 3x^2 - 2y^2 - 4z^2 + 54 & = 0 \\ 5x^2 - 3y^2 - 7z^2 + 74 & = 0 \end{cases}$$

is linear in x^2, y^2, and z^2. Proceeding by row reduction of the coeffecient matrix,

$$\begin{bmatrix} 3 & -2 & -4 & 54 \\ 5 & -3 & -7 & 74 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 & 54 \\ 0 & 1/3 & -1/3 & -16 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 & -4/3 & 18 \\ 0 & 1 & -1 & -48 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & -14 \\ 0 & 1 & -1 & -48 \end{bmatrix}$$

so the system of equations given in the problem is *linearly* equivalent to

$$\begin{cases} x^2 - 2z^2 - 14 & = 0 \\ y^2 - z^2 - 48 & = 0 \end{cases}$$

We will now work exclusively with the second equation. Since $y^2 - z^2 - 48 = 0$, $y^2 - z^2 = 48 = (y+z)(y-z)$. Since the solutions x, y, and z must be nonnegative integers, $(y+z)$ and $(y-z)$ must be integer factors of 48 and have the same sign. Now the factor $(y+z)$ is clearly positive if only nonnegative y and z are allowed (it must be nonzero because zero does not divide 48). Furthermore, $(y-z)$ is clearly less than or equal to $(y+z)$ for nonnegative y and z.

It has been shown that the factors $(y+z)$ and $(y-z)$ must be positive integers that multiply to 48 and $(y-z)$ must be the smaller factor unless both factors are equal. What are the possible values of the factors? All of the factor pairs of 48 with the larger factor listed first are $\{(8,6),(12,4),(16,3),(24,2),(48,1)\}$. The members of this set are nothing more than the ordered pairs $((y+z),(y-z))$.

To get values for y and z from these pairs, simply solve the system of equations

$$\begin{cases} y + z & = a \\ y - z & = b \end{cases}$$

for y and z. This system is trivially equivalent to

$$\begin{cases} (a+b)/2 & = y \\ (a-b)/2 & = z \end{cases}$$

The even-odd pairs $(16,3)$ and $(48,1)$ may be thrown out because they imply a noninteger y and z. Of the corresponding pairs left, the mapping $(a,b) \to ((a+b)/2,(a-b)/2)$ gives $(8,6) \to (7,1)$; $(12,4) \to (8,4)$; and $(24,2) \to (13,11)$.

We now know that any solution (x,y,z) of the original equations must be of the form $(x,7,1)$, $(x,8,4)$, or $(x,13,11)$. However, in the equivalent system, $x^2 = 2(z^2+7)$, so x is dependent on z. Since $2(4^2+7) = 46$ is not a perfect square, the form $(x,8,4)$ cannot be a solution for any integer x. Of the remaining pairs $(x,7,1)$ and $(x,13,11)$, the mapping $(x,y,z) \to (\sqrt{2(z^2+7)},y,z)$ gives $(4,7,1)$ and $(16,13,11)$.

Therefore, there are two nonnegative integer solutions: $(x,y,z) = (4,7,1)$ and $(x,y,z) = (16,13,11)$.